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Thermodynamical Theory of Galvanomagnetic and Thermomagnetic Phenomena.

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Chapter I.

GENERAL THEORY

1. – Thermodynamical Theory of Irreversible Processes.

The thermodynamical theory of irreversible processes [1,2] is a relatively new branch of science which enables us to treat real processes, from a macroscopic point of view, in a systematic way. The theory is based on two fundamental concepts, the *entropy production* and the *Onsager relations* [3], which are briefly introduced in this section.

The entropy production, which is a result of the progress of irreversible processes, can be calculated with the help of the fundamental laws of macroscopic physics: the law of conservation of mass, the equations of motion (momentum law), the first law of thermodynamics (energy conservation law) and the second law of thermodynamics (Gibbs' relation). From the fundamental laws a balance equation for the entropy can be derived:

$$\text{(I.1)} \qquad \partial s_v/\partial t = -\operatorname{div}(\boldsymbol{J}_s + s_v \boldsymbol{v}) + \sigma\,,$$

which shows that the entropy s_v (per unit volume) changes as a result of an *entropy flow* $\boldsymbol{J}_s$ and an *entropy production* σ. The divergence of the entropy flow gives the entropy supplied by exchange of energy or matter with the surroundings (it may be positive or negative), while σ arises from the action of irreversible processes inside the system. Calculations show that the entropy production has an expression of the following form:

$$\text{(I.2)} \qquad \sigma = \sum_i J_i X_i \geqslant 0\,,$$

where we have a sum of products of quantities J_i called « *fluxes* » (e.g., heat flow, diffusion flow, electric current, chemical reaction rate) and corresponding quantities X_i called « *forces* » or « *affinities* » (e.g., temperature gradient, gradient of chemical potential, electric field strength, chemical affinity). This expression is a measure of the irreversibility of the processes: it vanishes when the system undergoes only reversible changes, in all other cases it is positive.

In the most general case every flow is caused by contribution of all forces (and inversely). So in the linear approximation the irreversible phenomena can be expressed by the so-called « *phenomenological equations* », which have the following form:

$$\text{(I.3)} \qquad J_i = \sum_k L_{ik} X_k\,.$$

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(1) S. R. de Groot: *Thermodynamics of Irreversible Processes* (Amsterdam and New York, 1951).

(2) I. Prigogine: *Étude thermodynamique des phénomènes irréversibles* (Liege, 1947).

(3) L. Onsager: *Phys. Rev.*, 37, 405 (1931); 38, 2265 (1931).

Examples of these laws are, for instance, Fourier's law for the heat conduction, the law which describes the Dufour effect, Fick's law for the diffusion, the thermal diffusion (Soret-effect) law, Ohm's law for the electrical conduction, the laws which describe the thermoelectric effects and the laws which describe the galvanomagnetic and thermomagnetic effects.

The coefficients L_{ik} are called « *phenomenological coefficients* », and ONSAGER (3) has established the following « *reciprocal relations* » which hold between them:

$$L_{ik} = L_{ki}\,, \tag{I.4}$$

expressing a connection between reciprocal phenomena which arise from mutual interference of simultaneously occurring irreversible processes (*). Many well-known physical relations, which could not be explained by thermostatic considerations, or by considerations on the physical symmetry of the system, are a consequence of the « *Onsager relations* ». It is allowed to say that the *Onsager relations* are indeed a new and very fundamental law of macroscopic thermodynamics. ONSAGER has shown that they are the macroscopic counterpart of the property of time reversal invariance of the laws which govern the motion of the individual particles forming the system. The demonstration of ONSAGER is based on the general ideas of statistical mechanics, and it will be explained in a modified form in the next section.

2. – De Groot and Mazur's Theory.

2·1. *Casimir's objection and method.* – As CASIMIR (4) pointed out, ONSAGER's proof is strictly valid only for scalar phenomena. As a matter of fact, ONSAGER assumes that the irreversible fluxes can be considered time derivatives of thermodynamic state variables. This is correct for scalar processes (such as chemical reactions and relaxation phenomena) but not for vectorial processes, (such as heat conduction, diffusion and electrical conduction), and tensorial processes (such as viscous flow), and therefore an extension of the theory is necessary.

CASIMIR has proposed a method, which he has applied to the case of heat conduction in anisotropic crystals without magnetic field: he reformulates the problem under investigation in such a way that it is possible to apply the original Onsager formalism. To achieve this aim it is necessary to introduce appropriate auxiliary phenomenological equations containing an infinite number

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(*) In case there is a magnetic field $\boldsymbol{B}$, the phenomenological coefficients can be functions of $\boldsymbol{B}$ and then the relation (I.4) must be replaced by $L_{ik}(\boldsymbol{B}) = L_{ki}(-\boldsymbol{B})$.

(4) H. B. G. CASIMIR: *Rev. Mod. Phys.*, 17, 343 (1945).

of coefficients, which are only indirectly related to the ordinary macroscopic laws and coefficients. MAZUR and DE GROOT [5] have also considered systems with magnetic fields, and derived reciprocal relations for the heat and electrical conduction tensors.

2·2. *Basis of the method of De Groot and Mazur.* – To solve the problem of finding reciprocal relations for vectorial and tensorial processes, DE GROOT and MAZUR [6] have developed a different method, which is in a certain sense the opposite of the first: now the phenomenological problem is not rewritten in a form appropriate for the application of the original Onsager theory, but the formalism is generalized in such a way that it can be applied also for fluxes which describe vectorial and tensorial irreversible phenomena, or, in other words, for fluxes which are not necessarily time derivatives of state variables. In this theory the ordinary macroscopic laws can be used as phenomenological equations, and thus the complications inherent in the approach of the first method are avoided. With the help of a generalization of the fluctuation theory, (cf. also ([6-bis])), reciprocal relations for vectorial and tensorial processes can be derived directly from the property of microscopic reversibility.

2·3. *Fluctuation theory.* – We consider an energetically insulated system. The thermodynamic state of the system is described by variables $A_1, A_2, ..., A_m$, which are even functions of the particle velocities, and variables $B_{m+1}, B_{m+2}, ..., B_n$, which are odd functions of the particle velocities (the B-type variables, or their divergences, may be time derivatives of A-type variables: cf. also ([4])). All variables are continuous functions of space and time coordinates. For convenience the system will be divided into a number of cells of volume V^μ, in which the thermodynamic variables may be considered as uniform (here μ numbers the cells). The deviations of the state variables from their equilibrium values are denoted by

$$\alpha_i^\mu = A_i^\mu - (A_i^\mu)_{\text{equ}} , \qquad (i = 1, 2, ..., m) \tag{I.5}$$

$$\beta_k^\mu = B_k^\mu - (B_k^\mu)_{\text{equ}} . \qquad (k = m+1,\ m+2, ..., n) \tag{I.6}$$

Since the entropy has a maximum in the equilibrium state, the deviation of the entropy of the system from its equilibrium value is given, as a first approximation, by the quadratic form

$$\Delta S = -\tfrac{1}{2} \sum_{\mu,\nu} V^\mu V^\nu \left(\sum_{i,j} g_{ij}^{\mu\nu} \alpha_i^\mu \alpha_j^\nu + \sum_{k,l} h_{kl}^{\mu\nu} \beta_k^\mu \beta_l^\nu\right) , \tag{I.7}$$

where $g_{ij}^{\mu\nu}$ and $h_{kl}^{\mu\nu}$ are positive definite forms.

(5) P. MAZUR and S. R. DE GROOT: *Physica*, **19**, 961 (1953).

(6) S. R. DE GROOT and P. MAZUR: *Phys. Rev.*, **94**, 218 (1954); S. R. DE GROOT: *Supplemento al Nuovo Cimento*, **12**, 5 (1954).

(6-bis) J. VLIEGER and S. R. DE GROOT: *Physica*, **20**, 372 (1954).

Since the entropy is an even function of the particle velocities no cross-terms between α- and β-variables appear in the expression (I.7). The probability distribution for the α_i^μ and β_k^μ is expressed by

$$P \prod_{\mu,i,k} \mathrm{d}\alpha_i^\mu \, \mathrm{d}\beta_k^\mu = \frac{\exp[\Delta S/k] \prod_{\mu,i,k} \mathrm{d}\alpha_i^\mu \, \mathrm{d}\beta_k^\mu}{\int \dots \int \exp[\Delta S/k] \prod_{\mu,i,k} \mathrm{d}\alpha_i^\mu \mathrm{d}\beta_k^\mu}, \tag{I.8}$$

where k is Boltzmann's constant.

The following linear combinations of parameters are introduced:

$$X_i^\mu = (V^\mu)^{-1} \partial \Delta S / \partial \alpha_i^\mu = - \sum_\nu V^\nu \sum_j g_{ij}^{\mu\nu} \alpha_j^\nu , \tag{I.9}$$

$$y_k^\mu = (V^\mu)^{-1} \partial \Delta S / \partial \beta_k^\mu = - \sum_\nu V^\nu \sum_l h_{kl}^{\mu\nu} \beta_l^\nu , \tag{I.10}$$

which will be designed as conjugate variables.

The following averages can easily be found with the help of (I.8)-(I.10):

$$\overline{\alpha_i^\mu(t) X_j^\nu(t)} = - k \, \delta_{ij} \, \delta_{\mu\nu} / V^\mu , \tag{I.11}$$

$$\overline{\beta_k^\mu(t) y_l^\nu(t)} = - k \, \delta_{kl} \, \delta_{\mu\nu} / V^\mu , \tag{I.12}$$

where δ is the Kronecker symbol.

Here and in the following, according to the fundamental notions of statistical mechanics, the average may be interpreted either as an average over a microcanonical ensemble of systems, or as a time average for one single system.

Average values of products of α- and β-type variables always vanish.

Passing to the limit of continuous variables, the indices μ and ν of (I.11) and (I.12), which indicate the cells, can be replaced by $\boldsymbol{r}$ and $\boldsymbol{r}'$, which denote space coordinates, whereas the last two factors of (I.11) and (I.12) combine into a Heaviside-Dirac δ function. Consequently (I.11) and (I.12) become

$$\overline{\alpha_i(\boldsymbol{r}, t) X_j(\boldsymbol{r}', t)} = - k \, \delta_{ij} \, \delta(\boldsymbol{r} - \boldsymbol{r}') , \tag{I.13}$$

$$\overline{\beta_k(\boldsymbol{r}, t) y_l(\boldsymbol{r}', t)} = - k \, \delta_{kl} \, \delta(\boldsymbol{r} - \boldsymbol{r}') . \tag{I.14}$$

From (I.13) and (I.14) one can immediately derive the formulae

$$\overline{\alpha_i(\boldsymbol{r}, t) \Omega(\boldsymbol{r}') X_j(\boldsymbol{r}', t)} = - k \, \delta_{ij} \, \Omega(\boldsymbol{r}') \, \delta(\boldsymbol{r} - \boldsymbol{r}') , \tag{I.15}$$

$$\overline{\beta_k(\boldsymbol{r}, t) \Omega(\boldsymbol{r}') y_l(\boldsymbol{r}', t)} = - k \, \delta_{kl} \, \Omega(\boldsymbol{r}') \, \delta(\boldsymbol{r} - \boldsymbol{r}') , \tag{I.16}$$

where $\Omega(\boldsymbol{r})$ is a differential operator of the general form

$$\Omega(\boldsymbol{r}) = \sum_{p,q,s} a_{p,q,s}(\boldsymbol{r})\, \partial^{p+q+s}/\partial x_1^p \partial x_2^q \partial x_3^s \,. \tag{I.17}$$

The coefficients $a_{p,q,s}$ are independent of the state variables $\beta_k(\boldsymbol{r}, t)$ and $\alpha_i(\boldsymbol{r}, t)$; the Cartesian coordinates are denoted by x_1, x_2 and x_3.

The formulae (I.15) and (I.16) serve as an extremely useful basis for the straightforward derivation of the reciprocal relations among phenomenological coefficients.

Remark. It may be noted that with (I.9) and (I.10) the time derivative of (I.7) (entropy production per unit time) can be written as

$$\Delta\dot{S} = \sum_\mu V^\mu \left(\sum_i \dot{\alpha}_i^\mu X_i^\mu + \sum_k \dot{\beta}_k^\mu y_k^\mu\right), \tag{I.18}$$

or

$$\Delta\dot{S} = \int \left\{\sum_i \dot{\alpha}_i(\boldsymbol{r}) X_i(\boldsymbol{r}) + \sum_k \dot{\beta}_k(\boldsymbol{r}) y_k(\boldsymbol{r})\right\} \mathrm{d}\boldsymbol{r}\,, \tag{I.19}$$

in the limit of infinitely small cells. One sees that state variables and conjugate variables can be identified either from the expression for the deviation of the entropy, or from the expression for the entropy production (I.19).

2·4. *Microscopic reversibility.* – The fact that, on the average, the future behaviour of a system is identical with its past behaviour, will now be expressed with the help of correlation functions. Since a magnetic field $\boldsymbol{B}$ is present, it is necessary to reverse its direction in every point in order to have the particles retracing their paths. Minus sign arises when an α-type variable is combined with a β-type variable [4,1]:

$$\overline{\alpha_i(\boldsymbol{r}, t)\alpha_j(\boldsymbol{r}', t+\tau)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = \overline{\alpha_j(\boldsymbol{r}', t)\alpha_i(\boldsymbol{r}, t+\tau)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,, \tag{I.20}$$

$$\overline{\alpha_i(\boldsymbol{r}, t)\beta_l(\boldsymbol{r}', t+\tau)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = -\overline{\beta_l(\boldsymbol{r}', t)\alpha_i(\boldsymbol{r}, t+\tau)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,, \tag{I.21}$$

$$\overline{\beta_l(\boldsymbol{r}, t)\beta_m(\boldsymbol{r}', t+\tau)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = \overline{\beta_m(\boldsymbol{r}', t)\beta_l(\boldsymbol{r}, t+\tau)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,. \tag{I.22}$$

From these three relations it follows that

$$\overline{\alpha_i(\boldsymbol{r}, t)(\partial/\partial t)\alpha_j(\boldsymbol{r}', t)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = \overline{\alpha_j(\boldsymbol{r}', t)(\partial/\partial t)\alpha_i(\boldsymbol{r}, t)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,, \tag{I.23}$$

$$\overline{\alpha_i(\boldsymbol{r}, t)(\partial/\partial t)\beta_l(\boldsymbol{r}', t)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = -\overline{\beta_l(\boldsymbol{r}', t)(\partial/\partial t)\alpha_i(\boldsymbol{r}, t)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,, \tag{I.24}$$

$$\overline{\beta_l(\boldsymbol{r}, t)(\partial/\partial t)\beta_m(\boldsymbol{r}', t)}\,\{\boldsymbol{B}, \boldsymbol{B}'\} = \overline{\beta_m(\boldsymbol{r}', t)(\partial/\partial t)\beta_l(\boldsymbol{r}, t)}\,\{-\boldsymbol{B}, -\boldsymbol{B}'\}\,. \tag{I.25}$$

$\boldsymbol{B}$ and $\boldsymbol{B}'$ indicate the magnetic field strenghts at the positions $\boldsymbol{r}$ and $\boldsymbol{r}'$ where the averages are performed. The time derivative in the last three re-

lations should be interpreted on a microscopic scale as the average decay of the fluctuations, which is described by difference quotients [4,1]; however, for all practical macroscopic purposes this quotient can be considered as a real derivative.

2·5. *Regression of fluctuations.* – In order to find the reciprocal relations between the phenomenological coefficients, it is necessary to express the time derivatives of the state variables α_i and β_k as functions of the other state quantities χ_i and y_k or of their derivatives with respect to space coordinates. (As a matter of fact we shall write instead of the state variables, their deviations from the equilibrium value). That can be achieved with the use of the conservation laws and the entropy balance equations, and one obtains expressions of the kind

$$\partial\alpha_i/\partial t = \textstyle\sum_k \Omega_{ik}^{\alpha\alpha}\chi_k + \sum_h \Omega_{ih}^{\alpha\beta} y_h\,, \tag{I.26}$$

$$\partial\beta_i/\partial t = \textstyle\sum_k \Omega_{ik}^{\beta\alpha}\chi_k + \sum_h \Omega_{ih}^{\beta\beta} y_h\,. \tag{I.27}$$

The operators Ω_{ik} and Ω_{ih} turn out to be like the following: $\mathbf{L}_{ik}\cdot$, $\mathbf{L}_{ik}\cdot\mathrm{grad}$, div $(\mathbf{L}_{ik}\cdot)$, div $(\mathbf{L}_{ik}\cdot\mathrm{grad})$, etc. The subscripts indicate here the fluxes and the forces, not the Cartesian components).

2·6 *Derivation of the Onsager relations.* – With the preceding results, and the usual assumption that the average decay of fluctuations follows the phenomenological macroscopic laws, (which is true for linear phenomena, so that we can take for the average decay of fluctuations, which appears in formulae (I.22)–(I.23), the expressions (I.26) and (I.27)), the derivation of Onsager relations is straightforward. Introducing (I.26) and (I.27) into (I.23)–(I.25) one has

$$\begin{aligned}
&\overline{\alpha_i(\boldsymbol{r})\left\{\textstyle\sum_k \Omega_{jk}^{\alpha\alpha}(\boldsymbol{r}',\boldsymbol{B}')\ \chi_k(\boldsymbol{r}') + \sum_h \Omega_{jh}^{\alpha\beta}(\boldsymbol{r}',\boldsymbol{B}')\ y_h(\boldsymbol{r}')\right\}}\{\boldsymbol{B},\boldsymbol{B}'\} = \\
&\quad = \overline{\alpha_j(\boldsymbol{r}')\left\{\textstyle\sum_k \Omega_{ik}^{\alpha\alpha}(\boldsymbol{r},-\boldsymbol{B})\chi_k(\boldsymbol{r}) + \sum_h \Omega_{ih}^{\alpha\beta}(\boldsymbol{r},-\boldsymbol{B})y_h(\boldsymbol{r})\right\}}\{-\boldsymbol{B},-\boldsymbol{B}'\},
\end{aligned} \tag{I.28}$$

$$\begin{aligned}
&\overline{\alpha_i(\boldsymbol{r})\left\{\textstyle\sum_k \Omega_{lk}^{\beta\alpha}(\boldsymbol{r}',\boldsymbol{B}')\ \chi_k(\boldsymbol{r}') + \sum_h \Omega_{lh}^{\beta\beta}(\boldsymbol{r}',\boldsymbol{B}')y_h(\boldsymbol{r}')\right\}}\{\boldsymbol{B},\boldsymbol{B}'\} = \\
&\quad = -\overline{\beta_l(\boldsymbol{r}')\left\{\textstyle\sum_k \Omega_{ik}^{\alpha\alpha}(\boldsymbol{r},-\boldsymbol{B})\ \chi_k(\boldsymbol{r}) + \sum_h \Omega_{ih}^{\alpha\beta}(\boldsymbol{r},-\boldsymbol{B})y_h(\boldsymbol{r})\right\}}\{-\boldsymbol{B},-\boldsymbol{B}'\},
\end{aligned} \tag{I.29}$$

$$\begin{aligned}
&\overline{\beta_l(\boldsymbol{r})\left\{\textstyle\sum_k \Omega_{mk}^{\beta\alpha}(\boldsymbol{r}',\boldsymbol{B}')\ \chi_k(\boldsymbol{r}') + \sum_h \Omega_{mh}^{\beta\beta}(\boldsymbol{r}',\boldsymbol{B}')y_h(\boldsymbol{r}')\right\}}\{\boldsymbol{B},\boldsymbol{B}'\} = \\
&\quad = \overline{\beta_m(\boldsymbol{r}')\left\{\textstyle\sum_k \Omega_{lk}^{\beta\alpha}(\boldsymbol{r},-\boldsymbol{B})\chi_k(\boldsymbol{r}) + \sum_h \Omega_{lh}^{\beta\beta}(\boldsymbol{r},-\boldsymbol{B})\ y_h(\boldsymbol{r})\right\}}\{-\boldsymbol{B},-\boldsymbol{B}'\}.
\end{aligned} \tag{I.30}$$

According to (I.15) and (I.16) these relations lead to

$$(\text{I.31}) \qquad -k\delta_{ik}\Omega^{\alpha\alpha}_{jk}(\boldsymbol{r}', \boldsymbol{B}')\delta(\boldsymbol{r}-\boldsymbol{r}') = -k\delta_{jk}\Omega^{\alpha\alpha}_{ik}(\boldsymbol{r}, -\boldsymbol{B})\,\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$(\text{I.32}) \qquad -k\delta_{ik}\Omega^{\beta\alpha}_{ik}(\boldsymbol{r}', \boldsymbol{B}')\delta(\boldsymbol{r}-\boldsymbol{r}') = +k\delta_{ih}\Omega^{\alpha\beta}_{ih}(\boldsymbol{r}, -\boldsymbol{B})\delta(\boldsymbol{r}-\boldsymbol{r}')\,,$$

$$(\text{I.33}) \qquad -k\delta_{ih}\Omega^{\beta\beta}_{mh}(\boldsymbol{r}', \boldsymbol{B}')\delta(\boldsymbol{r}-\boldsymbol{r}') = -k\delta_{mh}\Omega^{\beta\beta}_{ih}(\boldsymbol{r}, -\boldsymbol{B})\delta(\boldsymbol{r}-\boldsymbol{r}')\,.$$

Elimination of the Kronecker δ's and the δ-functions gives now the relations for the phenomenological coefficients, which are contained into the operators Ω, either simply

$$(\text{I.34}) \qquad L_{ik}(\boldsymbol{B}) = L_{ki}(-\boldsymbol{B})\,,$$

or in a form containing differential operators, e.g.

$$(\text{I.35}) \qquad \operatorname{Div} L_{ik}(\boldsymbol{B}) = \operatorname{Div} L_{ki}(-\boldsymbol{B})\,,$$

according to the actual expression of Ω for the various cases.

Remark. Onsager's demonstration was valid for the case $J_i = \partial\alpha_i/\partial t$ and $X_i = X_i$. Reciprocal relations are then immediately obtained: since $\Omega_{ik} = L_{ik}$, and from (I.3), (I.23) and (I.15), follows $L_{ik} = L_{ki}$.

Chapter II.

DERIVATION OF THE RECIPROCAL RELATIONS

1. – Introduction.

In this chapter the general theory outlined above (6) is applied to the case of mixtures of n charged components in an electromagnetic field (7). In such a general case the irreversible phenomena of heat conduction, diffusion (including electric conduction) and cross-effects, and of viscous flow, occur.

First the macroscopic phenomenological theory of the case mentioned is developed: in § 2 the entropy production in local form is calculated from the fundamental laws; in § 3 the phenomenological equations are written down.

(7) R. Fieschi, S. R. de Groot, P. Mazur and J. Vlieger: *Physica*, 20, 67 (1954).

Subsequently the fluctuation theory is developed and the reciprocal relations for the phenomenological coefficients are obtained (§§ 4-10). In § 4 we derive an expression for the total entropy production in an energetically insulated system in terms of fluctuations of independent thermodynamic state variables; in § 5 fluctuation averages are calculated, and in the next section (§ 6) the microscopic time reversibility property is expressed by means of correlation functions. From the phenomenological equations, the conservation laws and the entropy balance, we derive in § 7 the expression for the regression of fluctuations of the state variables. The equation for the decay of the vector potential requires special attention. In §§ 8, 9 and 10 the reciprocal relations for the phenomenological coefficients are derived. It is found that one gets physical results from time reversibility relations, in which both even and odd variables occur (4).

The last part of the chapter (§§ 11 and 12) shows how the description of simpler phenomena can be obtained by specializing the results of the more general case. In particular the equations which describe the galvanomagnetic and thermomagnetic phenomena are considered.

2. – Entropy Production in Local Form.

Let us consider a mixture of n charged non-reacting components in an electromagnetic field and subject to external conservative (non-electrical) forces. Magnetization and polarization are neglected. The entropy production can be calculated from the fundamental laws of macroscopic physics (1,8) (cf. chap. I, § 2).

The law of conservation of mass. – As chemical reactions are not taken into consideration, this law can be written as follows for the component k:

$$\varrho \, \mathrm{d}c_k/\mathrm{d}t = -\operatorname{div}\boldsymbol{J}_k\,, \qquad (k = 1, 2, ..., n) \tag{II.1}$$

where ϱ is the total density, c_k the mass fraction of component k $(\sum_{k=1}^{n} c_k = 1)$ and $\boldsymbol{J}_k = \varrho_k(\boldsymbol{v}_k - \boldsymbol{v})$ the diffusion flow of k with respect to the centre of gravity motion ($\varrho_k = c_k\varrho$ is density of component k, $\boldsymbol{v}_k$ is velocity of k and $\boldsymbol{v}$ is barycentric velocity). From the definition of $\boldsymbol{J}_k$ it follows that $\sum_{k=1}^{n} \boldsymbol{J}_k = 0$. Furthermore we have

$$\varrho \, \mathrm{d}v/\mathrm{d}t = \operatorname{div} \boldsymbol{v}\,. \tag{II.2}$$

(8) P. Mazur and I. Prigogine: *Acad. Roy. Belg., Classe des Sciences, Mém.*, 28, fasc. 1 (1953).

Here $v = \varrho^{-1}$ is the specific volume. In (II.1) and (II.2) $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative with respect to the centre of mass motion:

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\mathrm{grad}\,. \tag{II.3}$$

With (II.3) the formulae (II.1) and (II.2) can also be written as

$$\partial\varrho_k/\partial t = -\,\mathrm{div}\,\varrho_k\,\boldsymbol{v} - \mathrm{div}\,\boldsymbol{J}_k\,, \tag{II.4}$$

$$\partial\varrho/\partial t = -\,\mathrm{div}\,\varrho\boldsymbol{v}\,. \tag{II.5}$$

The momentum law.

$$\varrho\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -\,\mathrm{Div}\,\mathbf{P}^{\dagger} + \sum_{k=1}^{n}\varrho_k\boldsymbol{F}_k^{(L)} + \sum_{k=1}^{n}\varrho_k\boldsymbol{F}_k\,. \tag{II.6}$$

$\boldsymbol{F}_k^{(L)}$ is the Lorentz electromagnetic force per unit mass, acting on component k:

$$\boldsymbol{F}_k^{(L)} = e_k\{\boldsymbol{E} + c^{-1}(\boldsymbol{v}_k\wedge\boldsymbol{B})\}, \qquad (k = 1, 2, \ldots, n) \tag{II.7}$$

where $\boldsymbol{E}$ is the electric field strength, $\boldsymbol{B}$ the magnetic induction, e_k the charge per unit mass of component k. $\boldsymbol{F}_k$ is an external (non-electrical) conservative force, also per unit mass of k:

$$\boldsymbol{F}_k = -\,\mathrm{grad}\,w_k\,;\quad \text{with}\quad \partial w_k/\partial t = 0\,, \qquad (k = 1, 2, \ldots, n) \tag{II.8}$$

where w_k is the potential energy per unit mass of k. $\mathbf{P}$ is the pressure tensor

$$\mathbf{P} = p\delta + \boldsymbol{\Pi}\,, \tag{II.9}$$

with δ the unit tensor, p the hydrostatic pressure and $\boldsymbol{\Pi}$ the viscous pressure tensor. The sign $\dagger$ means transposing of the Cartesian components of a tensor.

Introducing the total charge density

$$\varrho e = \sum_{k=1}^{n}\varrho_k e_k\,, \tag{II.10}$$

and the conduction current

$$\boldsymbol{i} = \sum_{k=1}^{n} e_k\boldsymbol{J}_k\,, \tag{II.11}$$

formula (II.6) can be written as

$$\varrho\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -\,\mathrm{Div}\,\mathbf{P}^{\dagger} + \varrho e\{\boldsymbol{E} + c^{-1}(\boldsymbol{v}\wedge\boldsymbol{B})\} + c^{-1}(\boldsymbol{i}\wedge\boldsymbol{B}) + \sum_{k=1}^{n}\varrho_k\boldsymbol{F}_k\,. \tag{II.12}$$

Scalar multiplication of (II.12) with $\boldsymbol{v}$ gives the balance of the kinetic energy per unit of mass $l = \frac{1}{2}\boldsymbol{v}^2$:

$$\varrho\, \mathrm{d}l/\mathrm{d}t = -\boldsymbol{v}\cdot \mathrm{Div}\,\mathbf{P}^{\dagger} + \varrho e\boldsymbol{v}\cdot\boldsymbol{E} - c^{-1}\boldsymbol{i}\cdot(\boldsymbol{v}\wedge\boldsymbol{B}) + \sum_{k=1}^{n}\varrho_k\boldsymbol{v}\cdot\boldsymbol{F}_k\,. \tag{II.13}$$

Using (II.3), (II.5) and (II.8), we can write (II.12) alternatively as

$$\partial\boldsymbol{g}/\partial t = -\,\mathrm{Div}\,(\boldsymbol{g}\boldsymbol{v} + \mathbf{P}^{\dagger}) - \varrho e\{\mathrm{grad}\,\varphi + c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\} + \\ + c^{-1}(\boldsymbol{i}\wedge\boldsymbol{B}) - \sum_{k=1}^{n}\varrho_k\,\mathrm{grad}\,w_k\,, \tag{II.14}$$

where $\boldsymbol{g} = \varrho\boldsymbol{v}$ is the momentum density; furthermore use has been made of

$$\boldsymbol{E} = -\,\mathrm{grad}\,\varphi - c^{-1}\,\partial\boldsymbol{A}/\partial t\,, \tag{II.15}$$

where φ is the scalar potential and $\boldsymbol{A}$ the vector potential.

Energy law. – The energy of the electromagnetic field per unit of volume is

$$f_v = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\,. \tag{II.16}$$

The balance equation of f_v is Poynting's law

$$\partial f_v/\partial t = -\,\mathrm{div}\,c(\boldsymbol{E}\wedge\boldsymbol{B}) - \varrho e\boldsymbol{v}\cdot\boldsymbol{E} - \boldsymbol{i}\cdot\boldsymbol{E}\,, \tag{II.17}$$

where $c(\boldsymbol{E}\wedge\boldsymbol{B})$ is the Poynting vector. With (II.2) and (II.3) formula (II.17) can also be written as

$$\varrho\,\mathrm{d}f/\mathrm{d}t = -\,\mathrm{div}\{c(\boldsymbol{E}\wedge\boldsymbol{B}) - f_v\boldsymbol{v}\} - \varrho e\boldsymbol{v}\cdot\boldsymbol{E} - \boldsymbol{i}\cdot\boldsymbol{E}\,, \tag{II.18}$$

where $f = f_v\,v$ is the electromagnetic energy per unit mass.

For the potential energy $w = \sum_{k=1}^{n} c_k w_k$ per unit mass, we have

$$\mathrm{d}w/\mathrm{d}t = -\,\mathrm{div}\,(\sum_{k=1}^{n} w_k\boldsymbol{J}_k) - \sum_{k=1}^{n}\varrho_k\boldsymbol{v}_k\cdot\boldsymbol{F}_k\,, \tag{II.19}$$

where (II.1), (II.3) and (II.8) have been applied.

The energy equation can be written as

$$\varrho\,\mathrm{d}e/\mathrm{d}t = -\,\mathrm{div}\,\boldsymbol{J}_e\,, \tag{II.20}$$

where e is the total energy per unit mass and $\boldsymbol{J}_e$ the energy flow. These are given by

(II.21) $$\mathrm{e} = u + l + f + w ,$$

(II.22) $$\boldsymbol{J}_e = \boldsymbol{J}_q + \boldsymbol{v}\cdot\mathbf{P} + c(\boldsymbol{E}\wedge\boldsymbol{B}) - f_v\boldsymbol{v} + \sum_{k=1}^{n} w_k\boldsymbol{J}_k ,$$

with u the internal energy per unit of mass. Equation (II.22) defines the heat flow $\boldsymbol{J}_q$. [The balance equation of $\mathrm{e}_v = \mathrm{e}\varrho$ can be written as

(II.23) $$\partial\mathrm{e}_v/\partial t = - \operatorname{div} \boldsymbol{J}_{\mathrm{e,\,tot}} = - \operatorname{div} (\boldsymbol{J}_e + \mathrm{e}_v\boldsymbol{v}) ,$$

with

(II.24) $$\boldsymbol{J}_{\mathrm{e,\,tot}} = u_v\boldsymbol{v} + l_v\boldsymbol{v} + w_v\boldsymbol{v} + \boldsymbol{J}_q + \boldsymbol{v}\cdot\mathbf{P} + c(\boldsymbol{E}\wedge\boldsymbol{B}) + \sum_{k=1}^{n} w_k\boldsymbol{J}_k ,$$

which follows from (II.22) and $\mathrm{e}_v = u_v + l_v + f_v + w_v$ (the index v means per unit volume). In the formula (II.24) the electromagnetic (« anti »)-convection term $-f_v\boldsymbol{v}$ has disappeared, in place of which one has the ordinary convection terms $u_v\boldsymbol{v}$, $l_v\boldsymbol{v}$ and $w_v\boldsymbol{v}$].

The first law of thermodynamics (internal energy balance) is found by deducting (II.13), (II. 18) and (II.19) from (II.20) (with (II.21) and (II.22)):

(II.25) $$\varrho\, \mathrm{d}u/\mathrm{d}t = - \operatorname{div} \boldsymbol{J}_q + \boldsymbol{i}\cdot\{\boldsymbol{E} + c^{-1}(\boldsymbol{v}\wedge\boldsymbol{B})\} + \sum_{k=1}^{n} \boldsymbol{F}_k\cdot\boldsymbol{J}_k - \mathbf{P} : \operatorname{Grad} \boldsymbol{v} .$$

Gibb's law.

(II.26) $$T\, \mathrm{d}s/\mathrm{d}t = \mathrm{d}u/\mathrm{d}t + p\, \mathrm{d}v/\mathrm{d}t - \sum_{k=1}^{n} \mu_k\, \mathrm{d}c_k/\mathrm{d}t ,$$

where T is the temperature, s the entropy per unit mass and μ_k the chemical potential per unit mass.

Entropy balance. – Introducing (II.1), (II.2) and (II.25) and using also (II.9) and (II.11), equation (II.26) can be written as a balance equation,

(II.27) $$\varrho\, \mathrm{d}s/\mathrm{d}t = - \operatorname{div} \boldsymbol{J}_s + \sigma ,$$

where $\boldsymbol{J}_s$ is the entropy flow

(II.28) $$\boldsymbol{J}_s = (\boldsymbol{J}_q - \sum_{k=1}^{n} \mu_k\boldsymbol{J}_k)/T ,$$

and where the entropy production σ follows from

$$(\text{II.29})\quad T\sigma = -\boldsymbol{J}_q\cdot(\text{grad } T)/T + \sum_{k=1}^{n} \boldsymbol{J}_k\cdot[e_k\{\boldsymbol{E} + c^{-1}(\boldsymbol{v}\wedge\boldsymbol{B})\} + \boldsymbol{F}_k - \\ - T\,\text{grad }(\mu_k/T)] - \boldsymbol{\Pi} : \text{Grad }\boldsymbol{v} \geqslant 0\,.$$

The three terms on the right-hand side of (II.29) are the results of entropy production due to heat conduction, diffusion and viscous flow respectively.

Hereunder we shall use another form of $T\sigma$, which follows from (II.8), (II, 15) and (II.28):

$$(\text{II.30})\quad T\sigma = -\boldsymbol{J}_s\cdot\text{grad } T - \sum_{k=1}^{n} \boldsymbol{J}_k\cdot\{\text{grad }\widetilde{\mu}_k + e_k c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\} - \\ - \boldsymbol{\Pi} : \text{Grad }\boldsymbol{v} \geqslant 0\,,$$

where the quantity

$$(\text{II.31})\qquad \widetilde{\mu}_k = \mu_k + e_k\varphi + w_k \qquad (k = 1, 2, ..., n)$$

has been introduced.

Finally from (II.3), (II.5) and (II.27) one finds

$$(\text{II.32})\qquad \partial s_v/\partial t = -\text{div }(s_v\boldsymbol{v} + \boldsymbol{J}_s) + \sigma\,,$$

where $s_v = \varrho s$ is the entropy per unit of volume.

Remark. From (II.30) (or (II.29)) one can see that, when the barycentric velocity vanishes, the magnetic part of the Lorentz force does not give any explicit contribution to the entropy production. It can however be shown (in an analogous way as it has been done by HOOYMAN and others [9] for the case of the Coriolis forces) that an implicit dependence may exist.

3. – Phenomenological Equations.

In (II.30) $T\sigma$ is seen to be a sum of the products of two sets of variables, viz. the first set $\boldsymbol{J}_s$, $\boldsymbol{J}_k$ and $\boldsymbol{\Pi}$, which are called « fluxes », and the second set $-\text{grad } T$, $-\{\text{grad }\widetilde{\mu}_k + e_k c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\}$ and $-\text{Grad }\boldsymbol{v}$, which are called « forces » or « affinities » (*) (cf. (I.2)). With these fluxes and forces we can

(9) G. J. HOYMAN, H. HOLTAN Jr., P. MAZUR and S. R. DE GROOT: *Physica*, 19, 1095 (1953).

(*) The parameter T which multiplies σ can be included in the phenomenological coefficients, hence it is not taken into account in the expression of the forces.

establish the «*phenomenological equations*» as the linear relations (cf. (I.3))

$$\text{(II.33)} \qquad \boldsymbol{J}_s = -\mathbf{L}_{ss}\cdot\operatorname{grad} T - \sum_{l=1}^{n} \mathbf{L}_{sl}\cdot\{\operatorname{grad}\tilde{\mu}_l + e_l c^{-1}(\partial \boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\}\,,$$

$$\text{(II.34)} \qquad \boldsymbol{J}_k = -\mathbf{L}_{ks}\cdot\operatorname{grad} T - \sum_{l=1}^{n} \mathbf{L}_{kl}\cdot\{\operatorname{grad}\tilde{\mu}_l + e_l c^{-1}(\partial \boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\}\,, \qquad (k = 1, 2, \dots, n)$$

$$\text{(II.35)} \qquad \Pi_{ij} = -\sum_{m,n=1}^{3} L_{ij,mn}(\operatorname{Grad}\boldsymbol{v})_{mn}\,, \qquad (i, j = 1, 2, 3)$$

(k and l denote chemical components, i, j, m and n Cartesian components). In these formulae cross-effects between fluxes and forces of different tensorial character are not taken into consideration, although such effects might exist in anisotropic media. The present formalism could of course be easily extended to such a case. (In isotropic media these cross-effects do not exist, according to the so-called Curie-principle).

The tensors $\mathbf{L}_{sl}$, $\mathbf{L}_{ks}$ and $\mathbf{L}_{kl}$ $(k, l=1, 2, \dots, n)$ are not independent, because the relation

$$\text{(II.36)} \qquad \sum_{k=1}^{n} \boldsymbol{J}_k = 0$$

holds. Equations containing independent coefficients, however, are easily obtained when the dependent flow $\boldsymbol{J}_n$ is eliminated from the expression of the entropy production. From (II.30) and (II.36) one has

$$\text{(II.37)} \qquad T\sigma = -\boldsymbol{J}_s\cdot\operatorname{grad} T - \sum_{k=1}^{n-1} \boldsymbol{J}_k\cdot\{\operatorname{grad}(\tilde{\mu}_k - \tilde{\mu}_n) + (e_k - e_n)c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\} - \boldsymbol{\Pi} : \operatorname{Grad}\boldsymbol{v}\,,$$

hence the phenomenological equations

$$\text{(II.38)} \qquad \boldsymbol{J}_s = -\mathbf{L}_{ss}\cdot\operatorname{grad} T - \sum_{l=1}^{n-1} \mathbf{L}_{sl}\cdot\{\operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n) - (e_l - e_n)c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\},$$

$$\text{(II.39)} \qquad \boldsymbol{J}_k = -\mathbf{L}_{ks}\cdot\operatorname{grad} T - \sum_{l=1}^{n-1} \mathbf{L}_{kl}\cdot\{\operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n) - (e_l - e_n)c^{-1}(\partial\boldsymbol{A}/\partial t - \boldsymbol{v}\wedge\boldsymbol{B})\}.$$

$$(k=1, 2, \dots, n-1)$$

By using (II.36) and by comparing the new sets of equations with equations (II.33) and (II.34) one can easily prove that the new coefficients coincide with those coefficients of the first set, which have the same subscripts, and it can be

shown that the following relations exist:

$$\text{(II. 40)} \qquad \sum_{k=1}^{n} \mathbf{L}_{ks} = 0\,, \quad \sum_{k=1}^{n} \mathbf{L}_{kl} = 0\,, \quad \sum_{k=1}^{n} \mathbf{L}_{sk} = 0\,, \quad \sum_{k=1}^{n} \mathbf{L}_{lk} = 0\,. \quad (l = 1, 2, \ldots, n)$$

We now consider the phenomenological equations (II.35). We shall take the case usually considered of a symmetric tensor $\mathbf{\Pi}$. Then in (II.30) only the symmetric part of the tensor Grad $\boldsymbol{v}$ is left. Therefore, instead of (II.35), we have

$$\text{(II.41)} \qquad \Pi_{(ij)} = -\sum_{m,n=1}^{3} L_{(ij),(mn)} (\text{Grad}\, \boldsymbol{v})_{(mn)}\,,$$

where the notations (ij) and (mn) indicate the symmetrical character in the indices between brackets. Consequently we are left with 36 phenomenological coefficients $L_{(ij),(mn)}$ instead of 81 coefficients $L_{ij,mn}$ from (II.35).

4. – Entropy Production in Terms of Fluctuations.

In order to apply the fluctuation theory one should know the fluctuation of the entropy (I.7) of the whole energetically insulated system, in terms of the fluctuations of the parameters (I.5), (I.6) which determine the thermodynamical state of the system, and of the variables conjugated to these parameters (I.9), (I.10). However one can see that state variables and conjugate quantities can as well be found from the expression (I.19) for the entropy production per unit time, which is generally used instead of the fluctuation of the entropy (*).

The change of the entropy per unit time dS/dt of the whole adiabatically insulated system in terms of fluctuations will be found [10,11] by integrating (II.30) over the volume of the system, using the balance equations for mass, momentum and entropy and applying the boundary conditions for the insulation of the system.

(*) The use of either expression is of course correct when no relation exists between the time derivative of a state variable and other state variables. When this is not the case, however, one can have, in general, different results from the two expressions. In the present case it can be seen from (II.4) and (II.11) that relations of the mentioned kind exist. Nevertheless, since these are relations between the time derivative of a variable which is an even function of the particle velocities (α-type variable) and variables which are odd functions of the particle velocities (β-type variables), no possibility of confusion may arise: this results from the fact that entropy is an even function of the particle velocities, and is expressed by quadratic terms containing either α-type variables or β-type variables separately ((I.7)).

(10) P. Mazur and S. R. de Groot: *Phys. Rev.*, **94**, 224 (1954).

(11) R. Fieschi, S. R. de Groot and P. Mazur: *Physica*, **20**, 67 (1954).

Integration of (II.30) gives

(II.42)
$$\int_V T\sigma \mathrm{d}V = -\int_V \Big\{ \boldsymbol{J}_s \cdot \mathrm{grad}\, \Delta T + \sum_{k=1}^{n} \boldsymbol{J}_k \cdot \mathrm{grad}\, \Delta\tilde{\mu}_k + \boldsymbol{\Pi} : \mathrm{Grad}\, \Delta \boldsymbol{v} + \Delta \boldsymbol{i}\cdot c^{-1}\, \partial \Delta \boldsymbol{A}/\partial t - c^{-1}\Delta \boldsymbol{i}\cdot(\Delta \boldsymbol{v} \wedge \boldsymbol{B})\Big\} \mathrm{d}V\,,$$

were we have written, instead of the variables T, $\tilde{\mu}_k$, $\boldsymbol{A}$, $\boldsymbol{v}$ and $\boldsymbol{i}$ their fluctuations around their equilibrium values (*) and where (II.11) has been used. Partial integration of the first three terms on the right hand side of (II.42), with the boundary conditions of vanishing heat flow and material flows, gives

(II.43)
$$\int_V T\sigma \mathrm{d}V = \int_V \Big\{ \Delta T \,\mathrm{div}\, \boldsymbol{J}_s + \sum_{k=1}^{n} \Delta\tilde{\mu}_k \,\mathrm{div}\, \boldsymbol{J}_k + \Delta \boldsymbol{v} \cdot \mathrm{Div}\, \boldsymbol{\Pi}^{\dagger} - \Delta \boldsymbol{i}\cdot c^{-1} \partial \Delta \boldsymbol{A}/\partial t + \Delta \boldsymbol{v}\cdot (c^{-1}\Delta \boldsymbol{i} \wedge \boldsymbol{B})\Big\} \mathrm{d}V\,.$$

Using (II.4) and (II.32), and neglecting third order terms, (II.43) can be rewritten as

(II.44)
$$T_0 \int_V \sigma \,\mathrm{d}V = -\int_V \Big\{ \Delta T\, (\partial \Delta s_v/\partial t + \mathrm{div}\, s_v \Delta \boldsymbol{v}) + \sum_{k=1}^{n} \Delta\tilde{\mu}_k (\partial \Delta \varrho_k/\partial t + \mathrm{div}\, \varrho_k \Delta \boldsymbol{v}) - \Delta \boldsymbol{v}\cdot \mathrm{Div}\, \boldsymbol{\Pi}^{\dagger} + \Delta \boldsymbol{i}\cdot c^{-1}\, \partial \Delta \boldsymbol{A}/\partial t - \Delta \boldsymbol{v}\cdot (c^{-1}\,\Delta \boldsymbol{i} \wedge \boldsymbol{B})\Big\} \mathrm{d}V\,,$$

or, integrating again by parts and using the boundary conditions,

(II.45)
$$T_0 \int_V \sigma \,\mathrm{d}V = -\int_V \Big\{ \Delta T\, \partial \Delta s_v/\partial t + \sum_{k=1}^{n} \Delta\tilde{\mu}_k\, \partial \Delta \varrho_k/\partial t + \Delta \boldsymbol{i}\cdot c^{-1}\, \partial \Delta \boldsymbol{A}/\partial t - \Delta \boldsymbol{v}\cdot \Big(s_v \,\mathrm{grad}\, \Delta T + \sum_{k=1}^{n} \varrho_k \,\mathrm{grad}\, \Delta\tilde{\mu}_k - c^{-1}\,\Delta \boldsymbol{i} \wedge \boldsymbol{B} + \mathrm{Div}\, \boldsymbol{\Pi}^{\dagger}\Big)\Big\} \mathrm{d}V\,.$$

Applying (II.9) and the relation

(II.46)
$$s_v \,\mathrm{grad}\, T + \sum_{k=1}^{n} \varrho_k \,\mathrm{grad}\, \tilde{\mu}_k = \mathrm{grad}\, p + \varrho e \,\mathrm{grad}\, \varphi + \sum_{k=1}^{n} \varrho_k \,\mathrm{grad}\, w_k\,,$$

which follows from the second law, the Gibbs-Duhem relations, and (II.31),

(*) We have put $\Delta T = T - T_0$, $\Delta\tilde{\mu}_k = \tilde{\mu}_k - \tilde{\mu}_{0,k}$, $\Delta \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{A}_0$, $\Delta \boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}$, and $\Delta \boldsymbol{i} = \boldsymbol{i} - \boldsymbol{i}_0$ ($T_0, \tilde{\mu}_{0,k}$, $\boldsymbol{A}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{i}_0$ being the equilibrium values), and have applied the equilibrium conditions, which require the constancy in time of all parameters, the uniformity of T_0 and $\tilde{\mu}_{0,k}$ and the vanishing of $\boldsymbol{v}_0$ and $\boldsymbol{i}_0$.

equation (II.45) can be written as

$$T_0 \int_V \sigma \, \mathrm{d}V = - \int_V \{\Delta T \, \partial \Delta s_v / \partial t + \sum_{k=1}^{n} \Delta \tilde{\mu}_k \, \partial \Delta \varrho_k / \partial t + \Delta \boldsymbol{i} \cdot c^{-1} \, \partial \Delta \boldsymbol{A} / \partial t - \tag{II.47}$$

$$- \Delta \boldsymbol{v} \cdot (\varrho e \operatorname{grad} \varphi + \sum_{k=1}^{n} \varrho_k \operatorname{grad} w_k - c^{-1} \Delta \boldsymbol{i} \wedge \boldsymbol{B} + \operatorname{Div} \mathbf{P}^{\dagger})\} \mathrm{d}V \, .$$

Applying (II.14) one has

$$T_0 \int_V \sigma \, \mathrm{d}V = \int_V \{\Delta T \, \partial \Delta s_v / \partial t + \sum_{k=1}^{n} \Delta \tilde{\mu}_k \, \partial \Delta \varrho_k / \partial t - \Delta \boldsymbol{i} \cdot c^{-1} \, \partial \Delta \boldsymbol{A} / \partial t + \tag{II.48}$$

$$+ \Delta \boldsymbol{v} \cdot \partial \Delta \boldsymbol{g} / \partial t + \varrho e \Delta \boldsymbol{v} \cdot c^{-1} \, \partial \Delta \boldsymbol{A} / \partial t\} \mathrm{d}V \, .$$

Putting

$$\Delta \tilde{\boldsymbol{g}} = \Delta \boldsymbol{g} + \varrho e c^{-1} \Delta \boldsymbol{A} \, , \tag{II.49}$$

(II.48) becomes finally

$$T_0 \int_V \sigma \, \mathrm{d}V = \tag{II.50}$$

$$= - \int_V (\Delta T \, \partial \Delta s_v / \partial t + \sum_{k=1}^{n} \Delta \tilde{\mu}_k \, \partial \Delta \varrho_k / \partial t + \Delta \boldsymbol{i} \cdot c^{-1} \, \partial \Delta \boldsymbol{A} / \partial t + \Delta \boldsymbol{v} \cdot \partial \Delta \tilde{\boldsymbol{g}} / \partial t) \, \mathrm{d}V .$$

For an adiabatically insulated system, as no entropy is supplied to the system by its surroundings (cf. (II.28)), we have, from the integral of (II.27) and using Gauss' theorem, for the change of the total entropy per unit time $\mathrm{d}S/\mathrm{d}t$:

$$\mathrm{d}S/\mathrm{d}t = \int_V \sigma \, \mathrm{d}V = \tag{II.51}$$

$$= - T_0^{-1} \int_V (\Delta T \, \partial \Delta s_v / \partial t + \sum_{k=1}^{n} \Delta \tilde{\mu}_k \, \partial \Delta \varrho_k / \partial t + \Delta \boldsymbol{i} \cdot c^{-1} \, \partial \Delta \boldsymbol{A} / \partial t + \Delta \boldsymbol{v} \cdot \partial \Delta \tilde{\boldsymbol{g}} / \partial t) \, \mathrm{d}V \, .$$

In order to apply fluctuation theory, the system must be energetically insulated, i.e. not only adiabatically, and it must be remarked that this condition for a system in which electromagnetic phenomena occur requires special attention. As it is shown in [5,10] the condition of energetical insulation does not change however the expression (II.51).

In fluctuation theory we strictly need $\mathrm{d}S/\mathrm{d}t$ expressed as a function of independent state variables. The state variables Δs_v, $\Delta \varrho_k$, $\Delta \boldsymbol{A}$ and $\Delta \tilde{\boldsymbol{g}}$ in (II.51) however are not all independent, because there exist $n+1$ relations (in the form of integral equations). These relations are the conservation of

mass of the separate components and the conservation of total energy. As can easily be shown, these relations do not affect our results (cf. also ([6-bis])).

It is clear that (II.51) has the form (I.19), hence the required form for the change of entropy of an energetically insulated system, which is appropriate for the application of fluctuation theory (cf. (I.15), (I.16)) and microscopic reversibility (cf. (I.23)-(I.25)). The variables Δs_v and $\Delta \varrho_k$ are of the α-type, whereas the components of $c^{-1}\Delta \boldsymbol{A}$ and $\Delta \tilde{\boldsymbol{g}}$ are β-type variables. (ΔT and $\Delta \tilde{\mu}_k$ are the corresponding X-type variables; the components of $\Delta \boldsymbol{i}$ and $\Delta \boldsymbol{v}$ are the corresponding Y-type variables).

5. – Fluctuations.

To these variables we apply the relations (I.15) or (I.16). This gives

$$\overline{\Delta s_v(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta T(\boldsymbol{r}')} = kT_0\,\Omega(\boldsymbol{r}')\,\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{II.52}$$

$$\overline{\Delta s_v(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta \tilde{\mu}_l(\boldsymbol{r}')} = 0\,, \qquad (l=1, 2, ..., n) \tag{II.53}$$

$$\overline{\Delta s_v(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta i_j(\boldsymbol{r}')} = 0\,, \qquad (j=1, 2, 3) \tag{II.54}$$

$$\overline{\Delta s_v(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta v_j(\boldsymbol{r}')} = 0\,, \qquad (j=1, 2, 3) \tag{II.55}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta T(\boldsymbol{r}')} = 0\,, \qquad (k=1, 2, ..., n) \tag{II.56}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta \tilde{\mu}_l(\boldsymbol{r}')} = kT_0\,\delta_{kl}\,\Omega(\boldsymbol{r}')\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \qquad (k, l=1, 2, ..., n) \tag{II.57}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta i_j(\boldsymbol{r}')} = 0\,, \qquad (k=1, 2, ..., n;\ j=1, 2, 3) \tag{II.58}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta v_j(\boldsymbol{r}')} = 0\,, \qquad (k=1, 2, ..., n;\ j=1, 2, 3) \tag{II.59}$$

$$\overline{\Delta A_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta T(\boldsymbol{r}')} = 0\,, \qquad (i=1, 2, 3) \tag{II.60}$$

$$\overline{\Delta A_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta \tilde{\mu}_l(\boldsymbol{r}')} = 0\,, \qquad (i=1, 2, 3;\ l=1, 2, ..., n) \tag{II.61}$$

$$\overline{\Delta A_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta i_j(\boldsymbol{r}')} = ckT_0\delta_{ij}\,\Omega(\boldsymbol{r}')\,\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad (i, j=1, 2, 3) \tag{II.62}$$

$$\overline{\Delta A_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta v_j(\boldsymbol{r}')} = 0\,, \qquad (i, j=1, 2, 3) \tag{II.63}$$

$$\overline{\Delta \tilde{g}_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta T(\boldsymbol{r}')} = 0\,, \qquad (i=1, 2, 3) \tag{II.64}$$

$$\overline{\Delta \tilde{g}_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta \tilde{\mu}_l(\boldsymbol{r}')} = 0\,, \qquad (i=1, 2, 3;\ l=1, 2, ..., n) \tag{II.65}$$

$$\overline{\Delta \tilde{g}_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta i_i(\boldsymbol{r}')} = 0\,, \qquad (i, j=1, 2, 3) \tag{II.66}$$

$$\overline{\Delta \tilde{g}_i(\boldsymbol{r})\Omega(\boldsymbol{r}')\Delta v_j(\boldsymbol{r}')} = kT_0\,\delta_{ij}\Omega(\boldsymbol{r}')\ \delta(\boldsymbol{r}-\boldsymbol{r}')\,, \qquad (i, j=1, 2, 3) \tag{II.67}$$

where k, l indicate chemical components and i, j Cartesian components. $\Omega(\boldsymbol{r})$ is an operator of the form (I.17).

The formulae (II.52)–(II.67) will be used in the calculation of the reciprocal relations between phenomenological coefficients.

6. – Microscopic Time Reversibility.

The equations of motion of the individual particles are invariant under time reversal, when also the external magnetic induction $\boldsymbol{B}_0(\boldsymbol{r})$ is reversed in every point. This property can be expressed with correlation functions by the following expressions, which are examples of relations (I.23)-(I.25):

$$\overline{\Delta s_v(\boldsymbol{r})\,(\partial/\partial t)\Delta s_v(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta s_v(\boldsymbol{r}')\,(\partial/\partial t)\Delta s_v(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.68}$$

$$\overline{\Delta s_v(\boldsymbol{r})\,(\partial/\partial t)\Delta \varrho_k(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta \varrho_k(\boldsymbol{r}')\,(\partial/\partial t)\Delta s_v(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.69}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\,(\partial/\partial t)\Delta \varrho_l(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta \varrho_l(\boldsymbol{r}')\,(\partial/\partial t)\Delta \varrho_k(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.70}$$

$$\overline{\Delta s_v(\boldsymbol{r})\,(\partial/\partial t)\Delta A_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = -\overline{\Delta A_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta s_v(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.71}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})(\partial/\partial t)\Delta A_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = -\overline{\Delta A_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta \varrho_k(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.72}$$

$$\overline{\Delta A_i(\boldsymbol{r})\,(\partial/\partial t)\Delta A_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta A_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta A_i(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.73}$$

$$\overline{\Delta \tilde{g}_i(\boldsymbol{r})\,(\partial/\partial t)\Delta \tilde{g}_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta \tilde{g}_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta \tilde{g}_i(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.74}$$

$$\overline{\Delta s_v(\boldsymbol{r})\,(\partial/\partial t)\Delta \tilde{g}_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = -\overline{\Delta \tilde{g}_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta s_v(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.75}$$

$$\overline{\Delta \varrho_k(\boldsymbol{r})\,(\partial/\partial t)\Delta \tilde{g}_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = -\overline{\Delta \tilde{g}_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta \varrho_k(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.76}$$

$$\overline{\Delta A_i(\boldsymbol{r})(\partial/\partial t)\Delta \tilde{g}_j(\boldsymbol{r}')}\,\{\boldsymbol{B}_0, \boldsymbol{B}_0'\} = \overline{\Delta \tilde{g}_j(\boldsymbol{r}')\,(\partial/\partial t)\Delta A_i(\boldsymbol{r})}\,\{-\boldsymbol{B}_0, -\boldsymbol{B}_0'\}\,, \tag{II.77}$$

with $k, l = 1, 2, ..., n$ and $i, j = 1, 2, 3$. The averages in the members of the equations (II.71), (II.72), (II.75) and (II.76) have opposite signs, the reason for this being, that Δs_v and $\Delta \varrho_k$ are even functions of the particle velocities, whereas $\Delta \boldsymbol{A}$ and $\Delta \tilde{\boldsymbol{g}}$ are odd functions of these quantities (cf. chapter I, §§ **3**, **4**).

7. – Regression of Fluctuations.

In order to apply fluctuation theory, it is necessary to derive expressions for the decay of fluctuations, i.e. to express the time derivatives of the state variables Δs_v, $\Delta \varrho_k$, $\Delta \boldsymbol{A}$ and $\Delta \tilde{\boldsymbol{g}}$, appearing in (II.51), as functions of the conjugate variables ΔT, $\Delta \tilde{\mu}_k$, $\Delta \boldsymbol{i}$ and $\Delta \boldsymbol{v}$ from (II.51).

In order to achieve this for $\partial \boldsymbol{A}/\partial t$, we cannot just use one of the equations (II.34) or (II.39), since this would not give $\partial \boldsymbol{A}/\partial t$ as a function of the conjugate

variables mentioned. We can, however, proceed as follows. Introducing (II.40) into formula (II.11), we get

$$\boldsymbol{i} = \sum_{k=1}^{n-1} (e_k - e_n) \boldsymbol{J}_k \,. \tag{II.78}$$

Inserting (II.39) into (II.78) and then solving $c^{-1}\,\partial \boldsymbol{A}/\partial t$, we find

$$c^{-1}\,\partial \boldsymbol{A}/\partial t = -\boldsymbol{\Lambda}_{es}\cdot \operatorname{grad} T - \sum_{l=1}^{n-1} \boldsymbol{\Lambda}_{el}\cdot \operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n) - \boldsymbol{\Lambda}_{ee}\cdot \boldsymbol{i} + c^{-1}(\boldsymbol{v}\wedge \boldsymbol{B}), \tag{II.79}$$

with the abbreviations

$$\boldsymbol{\Lambda}_{ee}^{-1} \equiv \sum_{k',k''=1}^{n-1} (e_{k'} - e_n)(e_{k''} - e_n)\mathbf{L}_{k'k''}\,, \tag{II.80}$$

$$\boldsymbol{\Lambda}_{es} \equiv \sum_{k'=1}^{n-1} (e_{k'} - e_n)\boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k's}\,, \tag{II.81}$$

$$\boldsymbol{\Lambda}_{el} \equiv \sum_{k'=1}^{n-1} (e_{k'} - e_n)\boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k'l}\,. \qquad (l = 1, 2, \dots, n-1) \tag{II.82}$$

($\boldsymbol{\Lambda}_{ee}^{-1}$ in (II.80) means the inverse tensor of $\boldsymbol{\Lambda}_{ee}$). So, in this way $\partial \boldsymbol{A}/\partial t$ has been found as a linear function of the proper variables.

Substituting (II.79) into (II.38) and (II.39), we obtain

$$\boldsymbol{J}_s = -\boldsymbol{\Lambda}_{ss}\cdot \operatorname{grad} T - \sum_{l=1}^{n-1} \boldsymbol{\Lambda}_{sl}\cdot \operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n) + \boldsymbol{\Lambda}_{se}\cdot \boldsymbol{i}\,, \tag{II.83}$$

$$\boldsymbol{J}_k = -\boldsymbol{\Lambda}_{ks}\cdot \operatorname{grad} T - \sum_{l=1}^{n-1} \boldsymbol{\Lambda}_{kl}\cdot \operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n) + \boldsymbol{\Lambda}_{ke}\cdot \boldsymbol{i}\,, \quad (k = 1, 2, \dots, n-1) \tag{II.84}$$

with the abbreviations

$$\boldsymbol{\Lambda}_{ss} \equiv \mathbf{L}_{ss} - \sum_{k',k''=1}^{n-1} (e_{k'} - e_n)(e_{k''} - e_n)\mathbf{L}_{sk'}\cdot \boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k''s}\,, \tag{II.85}$$

$$\boldsymbol{\Lambda}_{sl} \equiv \mathbf{L}_{sl} - \sum_{k',k''=1}^{n-1} (e_{k'} - e_n)(e_{k''} - e_n)\mathbf{L}_{sk'}\cdot \boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k''l}\,, \qquad (l = 1, 2, \dots, n-1) \tag{II.86}$$

$$\boldsymbol{\Lambda}_{se} \equiv \sum_{k'=1}^{n-1} (e_{k'} - e_n)\mathbf{L}_{sk'}\cdot \boldsymbol{\Lambda}_{ee}\,, \tag{II.87}$$

$$\boldsymbol{\Lambda}_{ks} \equiv \mathbf{L}_{ks} - \sum_{k',k''=1}^{n-1} (e_{k'} - e_n)(e_{k''} - e_n)\mathbf{L}_{kk'}\cdot \boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k''s}\,, \qquad (k = 1, 2, \dots, n-1) \tag{II.88}$$

$$\boldsymbol{\Lambda}_{kl} \equiv \mathbf{L}_{kl} - \sum_{k',k''=1}^{n-1} (e_{k'} - e_n)(e_{k''} - e_n)\mathbf{L}_{kk'}\cdot \boldsymbol{\Lambda}_{ee}\cdot \mathbf{L}_{k''l}\,, \qquad (k, l = 1, 2, \dots, n-1) \tag{II.89}$$

$$\boldsymbol{\Lambda}_{ke} \equiv \sum_{k'=1}^{n-1} (e_{k'} - e_n)\mathbf{L}_{kk'}\cdot \boldsymbol{\Lambda}_{ee}\,. \tag{II.90}$$

We need to express the tensors $\mathbf{L}$ explicitly in terms of the tensors $\mathbf{\Lambda}$. From (II.81), (II.82) and (II.85)–(II.90) we easily find

$$\mathbf{L}_{ss} = \mathbf{\Lambda}_{ss} + \mathbf{\Lambda}_{se}\cdot\mathbf{\Lambda}_{ee}^{-1}\cdot\mathbf{\Lambda}_{es}\,, \tag{II.91}$$

$$\mathbf{L}_{sl} = \mathbf{\Lambda}_{sl} + \mathbf{\Lambda}_{se}\cdot\mathbf{\Lambda}_{ee}^{-1}\cdot\mathbf{\Lambda}_{el}\,, \qquad (l = 1, 2, \ldots, n-1) \tag{II.92}$$

$$\mathbf{L}_{ks} = \mathbf{\Lambda}_{ks} + \mathbf{\Lambda}_{ke}\cdot\mathbf{\Lambda}_{ee}^{-1}\cdot\mathbf{\Lambda}_{es}\,, \qquad (k = 1, 2, \ldots, n-1) \tag{II.93}$$

$$\mathbf{L}_{kl} = \mathbf{\Lambda}_{kl} + \mathbf{\Lambda}_{ke}\cdot\mathbf{\Lambda}_{ee}^{-1}\cdot\mathbf{\Lambda}_{el}\,. \qquad (k, l = 1, 2, \ldots, n-1) \tag{II.94}$$

Since we have n^2 tensors $\mathbf{L}$ (i.e. $9n^2$ coefficients) and $(n+1)^2$ tensors $\mathbf{\Lambda}$ (i. e. $9(n+1)^2$ coefficients), there must exist $9(2n+1)$ relations between the $\mathbf{\Lambda}$'s.

We can now write the equations for the decay of fluctuations. Inserting (II.83) into (II.32) and neglecting second order terms in fluctuations, (which is in agreement with the fact, that we have neglected third order terms in equation (II.51)), we obtain

$$\partial \Delta s_v/\partial t = \operatorname{div}\,(\mathbf{\Lambda}_{ss}\cdot\operatorname{grad}\Delta T) + \sum_{l=1}^{n-1}\operatorname{div}\,\{\mathbf{\Lambda}_{sl}\cdot\operatorname{grad}\,(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n)\} - \\ - \operatorname{div}\,(\mathbf{\Lambda}_{se}\cdot\Delta\boldsymbol{i}) - \operatorname{div}\,(s_v^{(0)}\,\Delta\boldsymbol{v})\,, \tag{II.95}$$

where $s_v^{(0)}$ is the equilibrium value of s_v.

Introducing (II.84) into (II.4) (for $k = 1, 2, \ldots, n-1$), we find

$$\partial \Delta \varrho_k/\partial t = \operatorname{div}\,(\mathbf{\Lambda}_{ks}\cdot\operatorname{grad}\Delta T) + \sum_{l=1}^{n-1}\operatorname{div}\,\{\mathbf{\Lambda}_{kl}\cdot\operatorname{grad}\,(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n)\} - \\ - \operatorname{div}\,(\mathbf{\Lambda}_{ke}\cdot\Delta\boldsymbol{i}) - \operatorname{div}\,(\varrho_k^{(0)}\,\Delta\boldsymbol{v})\,, \qquad (k = 1, 2, \ldots, n-1) \tag{II.96}$$

with $\varrho_k^{(0)}$ the equilibrium value of ϱ_k. From (II.4), (II.40) and (II.84) we can also find an equation for $\partial\Delta\varrho_n/\partial t$ as a function of the variables ΔT, $\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n$, $\Delta\boldsymbol{i}$ and $\Delta\boldsymbol{v}$. As a calculation however shows, this equation will not provide any new results in the derivation of reciprocal relations (see also § 8), and therefore we have not written it explicitly.

Equation (II.79), written in terms of fluctuations, becomes

$$c^{-1}\,\partial\Delta\boldsymbol{A}/\partial t = -\mathbf{\Lambda}_{es}\cdot\operatorname{grad}\Delta T - \sum_{l=1}^{n-1}\mathbf{\Lambda}_{el}\cdot\operatorname{grad}\,(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n) - \\ - \mathbf{\Lambda}_{ee}\cdot\Delta\boldsymbol{i} + c^{-1}(\Delta\boldsymbol{v}\wedge\boldsymbol{B}_0)\,, \tag{II.97}$$

$\boldsymbol{B}_0$ being the external magnetic induction, which is supposed to be independent of the time.

Finally we find from (II.14), with (II.9), (II.41), (II.46) and (II.49)

$$\text{(II.98)}\qquad \partial\Delta\tilde{g}_i/\partial t = -\, s_v^{(0)}\,\partial\Delta T/\partial x_i - \sum_{k=1}^{n}\varrho_k^{(0)}\,\partial\Delta\tilde{\mu}_k/\partial x_i + c^{-1}(\Delta\boldsymbol{i}\wedge\boldsymbol{B}_0)_i +$$
$$+\,\varrho^{(0)}e^{(0)}c^{-1}(\Delta\boldsymbol{v}\wedge\boldsymbol{B}_0)_i +$$
$$+\sum_{u=1}^{3}(\partial/\partial x_u)\Big\{\sum_{m,n=1}^{3}L_{(ui)(mn)}\,\tfrac{1}{2}(\partial\Delta v_n/\partial x_m + \partial\Delta v_m/\partial x_n)\Big\}\,,\qquad (i=1,2,3)$$

where $\varrho^{(0)}$ and $e^{(0)}$ are the total density and the total charge per unit mass respectively, in the state of thermodynamic equilibrium, and where $i = 1, 2, 3$ denote Cartesian components.

Equations (II.95), (II.96), (II.97) and (II.98) are the desired equations, which express the time derivatives of Δs_v, $\Delta\varrho_k$, $\Delta\boldsymbol{A}$ and $\Delta\tilde{\boldsymbol{g}}$ in terms of ΔT, $\Delta\tilde{\mu}_k$, $\Delta\boldsymbol{i}$ and $\Delta\boldsymbol{v}$.

Remark. In general the tensors $\mathbf{L}$, $\boldsymbol{\Lambda}$ and the coefficients $L_{(ij)(mn)}$ will depend on $\boldsymbol{r}$ and $\boldsymbol{B}^*(\boldsymbol{r})$, where $\boldsymbol{B}^*(\boldsymbol{r})$ is the magnetic induction at the point $\boldsymbol{r}$, in a coordinate system, which moves with the same velocity as the (average) barycentric velocity $\boldsymbol{v}$ in that point. In fluctuation theory, however, the (average) barycentric velocity is zero, and therefore $\boldsymbol{B}^*(\boldsymbol{r})$ becomes $\boldsymbol{B}_0(\boldsymbol{r})$.

8. – Reciprocal Relations for Heat Conduction, Diffusion and Cross-effects.

The reciprocal relations can now easily be obtained by the preceding results (cf. chapter 1, § 5-6). One should insert the time derivatives of the state variables, expressed by the phenomenological laws (II.95)-(II.98), into the equations which express the macroscopic reversibility, and then use the results of the fluctuation theory.

We shall now derive reciprocal relations from the expressions (II.68)–(II.73) for microscopic reversibility. Substitution of (II.95) and (II.96) into (II.68), (II.69), and (II.70) gives:

$$\text{(II.99)}\qquad \overline{\Delta s_v[\mathrm{div}'\{\boldsymbol{\Lambda}'_{ss}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'\Delta T'\} + \sum_{l=1}^{n-1}\mathrm{div}'\{\boldsymbol{\Lambda}'_{sl}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'(\Delta\tilde{\mu}'_l - \Delta\tilde{\mu}'_n)\} -}$$
$$\overline{-\,\mathrm{div}'\{\boldsymbol{\Lambda}'_{se}(\boldsymbol{B}'_0)\cdot\Delta\boldsymbol{i}'\} - \mathrm{div}'(s_v^{(0)}\Delta\boldsymbol{v}')]} =$$
$$= \overline{\Delta s'_v[\mathrm{div}\{\boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\Delta T\} + \sum_{l=1}^{n-1}\mathrm{div}\{\boldsymbol{\Lambda}_{sl}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\,(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n)\} -}$$
$$\overline{-\,\mathrm{div}\{\boldsymbol{\Lambda}_{se}(-\,\boldsymbol{B}_0)\cdot\Delta\boldsymbol{i}\} - \mathrm{div}(s_v^{(0)}\Delta\boldsymbol{v})]}\,,$$

(II.100)
$$\Delta s_r \overline{[\mathrm{div}'\{\boldsymbol{\Lambda}'_{ks}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'\Delta T'\} + \sum_{l=1}^{n-1}\mathrm{div}'\{\boldsymbol{\Lambda}'_{kl}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'(\Delta\tilde{\mu}'_l - \Delta\tilde{\mu}'_n)\} -}$$
$$\overline{- \mathrm{div}'\{\boldsymbol{\Lambda}'_{ke}(\boldsymbol{B}'_0)\cdot\Delta \boldsymbol{i}'\} - \mathrm{div}'(\varrho_k^{(0)}\Delta \boldsymbol{v}')]} =$$
$$= \Delta\varrho'_k \overline{[\mathrm{div}\{\boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\Delta T\} + \sum_{l=1}^{n-1}\mathrm{div}\{\boldsymbol{\Lambda}_{sl}(-\boldsymbol{B}_0)\cdot\mathrm{grad}(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n)\} -}$$
$$\overline{- \mathrm{div}\{\boldsymbol{\Lambda}_{se}(-\boldsymbol{B}_0)\cdot\Delta \boldsymbol{i}\} - \mathrm{div}(s_v^{(0)}\Delta \boldsymbol{v})]}, \qquad (k = 1, 2, \ldots, n-1)$$

(II.101)
$$\Delta\varrho_k \overline{[\mathrm{div}'\{\boldsymbol{\Lambda}'_{ls}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'\Delta T'\} + \sum_{l'=1}^{n-1}\mathrm{div}'\{\boldsymbol{\Lambda}'_{ll'}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'(\Delta\tilde{\mu}'_{l'} - \Delta\tilde{\mu}'_n)\}}$$
$$\overline{- \mathrm{div}'\{\boldsymbol{\Lambda}_{le}(\boldsymbol{B}'_0)\cdot\Delta \boldsymbol{i}'\} - \mathrm{div}'(\varrho_l^{(0)}\Delta \boldsymbol{v}')]} =$$
$$= \Delta\varrho'_l \overline{[\mathrm{div}\{\boldsymbol{\Lambda}_{ks}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\Delta T\} + \sum_{k'=1}^{n-1}\mathrm{div}\{\boldsymbol{\Lambda}_{kk'}(-\boldsymbol{B}_0)\cdot\mathrm{grad}(\Delta\tilde{\mu}_{k'} - \Delta\tilde{\mu}_n)\} -}$$
$$\overline{- \mathrm{div}\{\boldsymbol{\Lambda}_{ke}(-\boldsymbol{B}_0)\cdot\Delta \boldsymbol{i}\} - \mathrm{div}(\varrho_k^{(0)}\Delta \boldsymbol{v})]}, \qquad (k, l = 1, 2, \ldots, n-1)$$

where dashes indicate dependence on $\boldsymbol{r}'$.

We also could have taken (II.68), (II.69) and (II.70) for k and (or) l equal to n. Then we should have to substitute an expression for $\partial\Delta\varrho_n/\partial t$ similar to (II.96). As it is already remarked in § 7, we do not get new results in this way. Therefore we have only taken formulae with $k, l = 1, 2, \ldots, n-1$. (For the same reason this is done below).

By means of (II.52)–(II.59), the equations (II.99)–(II.101) can be written as

(II.102)
$$\mathrm{div}'\{\boldsymbol{\Lambda}'_{ss}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\} = \mathrm{div}\{\boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\},$$

(II.103)
$$\mathrm{div}'\{\boldsymbol{\Lambda}'_{ks}(\boldsymbol{B}'_0)\cdot\mathrm{grad}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\} = \mathrm{div}\Big\{\sum_{l=1}^{n-1}\boldsymbol{\Lambda}_{sl}(-\boldsymbol{B}_0)(\delta_{kl}-\delta_{kn})\cdot$$
$$\cdot\mathrm{grad}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big\}, \qquad (k = 1, 2, \ldots, n-1)$$

(II.104)
$$\mathrm{div}'\Big\{\sum_{l'=1}^{n-1}\boldsymbol{\Lambda}'_{ll'}(\boldsymbol{B}'_0)\cdot(\delta_{kl'}-\delta_{kn})\,\mathrm{grad}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big\} =$$
$$= \mathrm{div}\Big\{\sum_{k'=1}^{n-1}\boldsymbol{\Lambda}_{kk'}(-\boldsymbol{B}_0)\cdot(\delta_{lk'}-\delta_{ln})\,\mathrm{grad}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big\}. \quad (k = 1, 2, \ldots, n-1)$$

The δ-functions can be eliminated by multiplying with an arbitrary function $f(\boldsymbol{r})$ and integrating over $\boldsymbol{r}$. From (II.102) we obtain

(II.105)
$$\mathrm{div}\{\boldsymbol{\Lambda}^{\dagger}_{ss}(\boldsymbol{B}_0)\cdot\mathrm{grad}\,f(\boldsymbol{r})\} = \mathrm{div}\{\boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)\cdot\mathrm{grad}\,f(\boldsymbol{r})\},$$

where Kronecker δ's have also been eliminated, and the left hand side of the equation has been twice partially integrated. The symbol $\dagger$ indicates transposing of the Cartesian components. Since $f(\boldsymbol{r})$ is an arbitrary function, we obtain, equating the coefficients of the same first order derivative of $f(\boldsymbol{r})$,

$$\text{(II.106)} \qquad \operatorname{Div} \boldsymbol{\Lambda}_{ss}(\boldsymbol{B}_0) = \operatorname{Div} \boldsymbol{\Lambda}^{\dagger}_{ss}(-\boldsymbol{B}_0)\,.$$

Moreover, equating in (II.105) the coefficients of the same second order derivative of $f(\boldsymbol{r})$ we have

$$\text{(II.107)} \qquad \boldsymbol{\Lambda}_{ss}(\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{ss}(\boldsymbol{B}_0) = \boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{ss}(-\boldsymbol{B}_0)\,.$$

In an analogous way we obtain, from (II.103) and (II.104),

$$\text{(II.108)} \qquad \begin{cases} \boldsymbol{\Lambda}_{sk}(\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{sk}(\boldsymbol{B}_0) = \boldsymbol{\Lambda}_{ks}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{ks}(-\boldsymbol{B}_0) & \text{and} \\ \operatorname{Div} \boldsymbol{\Lambda}_{sk}(\boldsymbol{B}_0) = \operatorname{Div} \boldsymbol{\Lambda}^{\dagger}_{ks}(-\boldsymbol{B}_0)\,, & (k = 1, 2, \dots, n-1) \end{cases}$$

$$\text{(II.109)} \qquad \begin{cases} \boldsymbol{\Lambda}_{kl}(\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{kl}(\boldsymbol{B}_0) = \boldsymbol{\Lambda}_{lk}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{lk}(-\boldsymbol{B}_0) & \text{and} \\ \operatorname{Div} \boldsymbol{\Lambda}_{kl}(\boldsymbol{B}_0) = \operatorname{Div} \boldsymbol{\Lambda}^{\dagger}_{lk}(-\boldsymbol{B}_0)\,. & (k, l = 1, 2, \dots, n-1) \end{cases}$$

Substitution of (II.95)–(II.97) into (II.71)–(II.73) gives:

$$\begin{aligned} \text{(II.110)} \qquad & \overline{c\,\Delta s_v\Big[-\boldsymbol{\Lambda}'_{es}(\boldsymbol{B}'_0)\cdot \operatorname{grad}'\Delta T' - \sum_{l=1}^{n-1} \boldsymbol{\Lambda}'_{el}(\boldsymbol{B}'_0)\cdot \operatorname{grad}'\,(\Delta\widetilde{\mu}'_l - \Delta\widetilde{\mu}'_n) -} \\ & \overline{-\boldsymbol{\Lambda}'_{ee}(\boldsymbol{B}'_0)\cdot \Delta\boldsymbol{i}' + c^{-1}(\Delta\boldsymbol{v}' \wedge \boldsymbol{B}'_0)\Big]_j} = -\overline{\Delta A'_j[\operatorname{div}\{\boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)\cdot \operatorname{grad}\Delta T\} +} \\ & \overline{+\sum_{l=1}^{n-1} \operatorname{div}\{\boldsymbol{\Lambda}_{sl}(-\boldsymbol{B}_0)\cdot \operatorname{grad}(\Delta\widetilde{\mu}_l - \Delta\widetilde{\mu}_n)\} - \operatorname{div}\{\boldsymbol{\Lambda}_{se}(-\boldsymbol{B}_0)\cdot \Delta\boldsymbol{i}\} - \operatorname{div}(s_v^{(0)}\Delta\boldsymbol{v})}, \\ & \hfill (j = 1, 2, 3) \end{aligned}$$

$$\begin{aligned} \text{(II.111)} \qquad & \overline{c\,\Delta\varrho_k\Big[-\boldsymbol{\Lambda}'_{es}(\boldsymbol{B}'_0)\cdot \operatorname{grad}'\,\Delta T' - \sum_{l=1}^{n-1} \boldsymbol{\Lambda}'_{el}(\boldsymbol{B}'_0)\cdot \operatorname{grad}'\,(\Delta\widetilde{\mu}'_l - \Delta\widetilde{\mu}'_n) -} \\ & \overline{-\boldsymbol{\Lambda}'_{ee}(\boldsymbol{B}'_0)\cdot \Delta\boldsymbol{i}' + c^{-1}(\Delta\boldsymbol{v}' \wedge \boldsymbol{B}'_0)\Big]_j} = -\overline{\Delta A_j(\boldsymbol{r}')[\operatorname{div}\{\boldsymbol{\Lambda}_{ks}(-\boldsymbol{B}_0)\cdot \operatorname{grad}\Delta T\} +} \\ & \overline{+\sum_{l=1}^{n-1} \operatorname{div}\{\boldsymbol{\Lambda}_{kl}(-\boldsymbol{B}_0)\cdot \operatorname{grad}(\Delta\widetilde{\mu}_l - \Delta\widetilde{\mu}_n)\} - \operatorname{div}\{\boldsymbol{\Lambda}_{ke}(-\boldsymbol{B}_0)\cdot \Delta\boldsymbol{i}\} - \operatorname{div}(\varrho_k^{(0)}\Delta\boldsymbol{v})}, \\ & (k = 1, 2, \dots, n-1;\ j = 1, 2, 3) \end{aligned}$$

$$\text{(II.112)} \qquad \overline{c\,\Delta A_i[-\mathbf{\Lambda}'_{es}(\boldsymbol{B}'_0)\cdot\operatorname{grad}'\Delta T' - \sum_{l=1}^{n-1}\mathbf{\Lambda}'_{el}(\boldsymbol{B}'_0)\cdot\operatorname{grad}'(\Delta\tilde{\mu}'_l - \Delta\tilde{\mu}'_n) -}$$
$$\overline{-\mathbf{\Lambda}'_{ee}(\boldsymbol{B}'_0)\cdot\Delta\boldsymbol{i}' + c^{-1}(\Delta\boldsymbol{v}'\wedge\boldsymbol{B}'_0)]_j} = \overline{c\,\Delta A'_j[-\mathbf{\Lambda}_{es}(-\boldsymbol{B}_0)\cdot\operatorname{grad}\Delta T -}$$
$$\overline{-\sum_{l=1}^{n-1}\mathbf{\Lambda}_{el}(-\boldsymbol{B}_0)\cdot\operatorname{grad}(\Delta\tilde{\mu}_l - \Delta\tilde{\mu}_n) - \mathbf{\Lambda}_{ee}(-\boldsymbol{B}_0)\cdot\Delta\boldsymbol{i} - c^{-1}(\Delta\boldsymbol{v}\wedge\boldsymbol{B}_0)]_i}\,,$$
$$(i, j = 1, 2, 3)$$

where the indices i and j at square brackets denote the components of the vector between the brackets.

The equations (II.110)–(II.112) can be written, with (II.52)–(II.63), as

$$\text{(II.113)} \qquad \sum_{u=1}^{3} \Lambda'_{es,ju}(\boldsymbol{B}'_0)(\partial/\partial x'_u)\,\delta(\boldsymbol{r}-\boldsymbol{r}') =$$
$$= -\sum_{u,i=1}^{3}(\partial/\partial x_u)\{\Lambda_{se,ui}(-\boldsymbol{B}_0)\,\delta_{ij}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\}\,, \qquad (j = 1, 2, 3)$$

$$\text{(II.114)} \qquad \sum_{l=1}^{n-1}\sum_{u=1}^{3} \Lambda'_{el,ju}(\boldsymbol{B}'_0)(\delta_{kl}-\delta_{kn})(\partial/\partial x'_u)\,\delta(\boldsymbol{r}-\boldsymbol{r}') =$$
$$= -\sum_{u,i=1}^{3}(\partial/\partial x_u)\{\Lambda_{ke,ui}(-\boldsymbol{B}_0)\,\delta_{ij}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\}\,, \quad (k = 1, 2, \ldots, n-1;\ j = 1, 2, 3)$$

$$\text{(II.115)} \qquad \sum_{u=1}^{3} \Lambda'_{ee,iu}(\boldsymbol{B}'_0)\,\delta_{iu}\,\delta(\boldsymbol{r}-\boldsymbol{r}') = \sum_{u=1}^{3}\Lambda_{ee,iu}(-\boldsymbol{B}_0)\,\delta_{ju}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \quad (i\,j = 1, 2, 3)$$

where i, j and u denote Cartesian components. Eliminating again Kronecker δ's and δ-functions, we get

$$\text{(II.116)} \qquad \Lambda_{es,ju}(\boldsymbol{B}_0) = \Lambda_{se,uj}(-\boldsymbol{B}_0)\,, \quad \text{or:} \quad \mathbf{\Lambda}_{es}(\boldsymbol{B}_0) = \mathbf{\Lambda}^{\dagger}_{se}(-\boldsymbol{B}_0)\,,$$

$$\text{(II.117)} \qquad \Lambda_{ek,ju}(\boldsymbol{B}_0) = \Lambda_{ke,uj}(-\boldsymbol{B}_0)\,, \quad \text{or:} \quad \mathbf{\Lambda}_{ek}(\boldsymbol{B}_0) = \mathbf{\Lambda}^{\dagger}_{ke}(-\boldsymbol{B}_0)\,,$$

$$\text{(II.118)} \qquad \Lambda_{ee,ji}(\boldsymbol{B}_0) = \Lambda_{ee,ij}(-\boldsymbol{B}_0)\,, \quad \text{or:} \quad \mathbf{\Lambda}_{ee}(\boldsymbol{B}_0) = \mathbf{\Lambda}^{\dagger}_{ee}(-\boldsymbol{B}_0)\,,$$

with $i, j, u = 1, 2, 3$ and $k = 1, 2, \ldots, n-1$.

The relations (II.116) and (II.117) are examples of reciprocal relations, which follow from expressions for the microscopic time reversibility relations containing both even and odd variables ([4]).

The reciprocal relations (II.107)–(II.109) and (II.116)–(II.118) have to be considered as intermediate results, because the $\mathbf{\Lambda}$'s have only been introduced as convenient abbreviations in fluctuation theory. The desired reciprocal relations for $\mathbf{L}_{ss}$, $\mathbf{L}_{sl}$, $\mathbf{L}_{ks}$ and $\mathbf{L}_{kl}$ can be found from equations (II.91)–(II.94)

with (II.107)–(II.109) and (II.116)–(II.118). From (II.91), (II.107), (II.116) and (II.118) follows

$$
\begin{aligned}
\text{(II.119)}\quad \mathbf{L}_{ss}(\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{ss}(\boldsymbol{B}_0) &= [\boldsymbol{\Lambda}_{ss}(\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{ss}(\boldsymbol{B}_0)] + [\boldsymbol{\Lambda}_{se}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1}_{ee}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}_{es}(\boldsymbol{B}_0) + \\
&\quad + \{\boldsymbol{\Lambda}_{se}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1}_{ee}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}_{es}(\boldsymbol{B}_0)\}^{\dagger}] = [\boldsymbol{\Lambda}^{\dagger}_{ss}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)] + \\
&\quad + [\boldsymbol{\Lambda}^{\dagger}_{es}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1\dagger}_{ee}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{\dagger}_{se}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}^{\dagger}_{es}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1\dagger}_{ee}(\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{\dagger}_{se}(\boldsymbol{B}_0)] = \\
&= [\boldsymbol{\Lambda}^{\dagger}_{ss}(-\boldsymbol{B}_0) + \boldsymbol{\Lambda}_{ss}(-\boldsymbol{B}_0)] + [\{\boldsymbol{\Lambda}_{se}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1}_{ee}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}_{es}(-\boldsymbol{B}_0)\}^{\dagger} + \\
&\quad + \boldsymbol{\Lambda}_{se}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}^{-1}_{ee}(-\boldsymbol{B}_0)\cdot\boldsymbol{\Lambda}_{es}(-\boldsymbol{B}_0)] = \mathbf{L}_{ss}(-\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{ss}(-\boldsymbol{B}_0)\,,
\end{aligned}
$$

and, in an analogous way,

$$\text{(II.120)}\qquad \operatorname{Div}\mathbf{L}_{ss}(\boldsymbol{B}_0) = \operatorname{Div}\mathbf{L}^{\dagger}_{ss}(-\boldsymbol{B}_0)\,.$$

In the same way it is possible to derive from (II.92), (II.93), (II.108) and (II.116)–(II.118)

$$
\text{(II.121)}\qquad \begin{cases} \mathbf{L}_{sk}(\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{sk}(\boldsymbol{B}_0) = \mathbf{L}_{ks}(-\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{ks}(-\boldsymbol{B}_0) & \text{and} \\ \operatorname{Div}\mathbf{L}_{sk}(\boldsymbol{B}_0) = \operatorname{Div}\mathbf{L}^{\dagger}_{ks}(-\boldsymbol{B}_0)\,, & (k = 1, 2, \ldots, n-1) \end{cases}
$$

and from (II.94), (II.109), (II.117) and (II.118)

$$
\text{(II.122)}\qquad \begin{cases} \mathbf{L}_{kl}(\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{kl}(\boldsymbol{B}_0) = \mathbf{L}_{lk}(-\boldsymbol{B}_0) + \mathbf{L}^{\dagger}_{lk}(-\boldsymbol{B}_0 & \text{and} \\ \operatorname{Div}\mathbf{L}_{kl}(\boldsymbol{B}_0) = \operatorname{Div}\mathbf{L}^{\dagger}_{lk}(-\boldsymbol{B}_0)\,. & (k, l = 1, 2, \ldots, n-1) \end{cases}
$$

With the convention of taking all phenomenological coefficients zero in empty space, and the fact that these coefficients do not depend on the shape of the sample it follows from (II.119)–(II.122)

$$\text{(II.123)}\qquad \mathbf{L}_{ss}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{ss}(-\boldsymbol{B}_0)\,,$$

$$\text{(II.124)}\qquad \mathbf{L}_{sk}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{ks}(-\boldsymbol{B}_0)\,, \qquad (k = 1, 2, \ldots, n-1)$$

$$\text{(II.125)}\qquad \mathbf{L}_{kl}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{lk}(-\boldsymbol{B}_0)\,. \qquad (k, l = 1, 2, \ldots, n-1)$$

Taking into account (II.36) and (II.37), relations (II.124) and (II.125) are also valid for k and l, or both, equal to n.

Equation (II.123)–(II.125) are the reciprocal relations for heat conduction, diffusion and cross-effects in anisotropic media in the presence of an electromagnetic field.

9. – Reciprocal Relations for Viscosity.

Here we shall consider the microscopic time reversibilities (II.74)–(II.77). Inserting equation (II.98) into (II.74), and using (II.64)–(II.66), we obtain

$$\text{(II.126)}\qquad \overline{\Delta\tilde{g}_i\big[\varrho^{(0)}e^{(0)}c^{-1}(\Delta\boldsymbol{v}'\wedge\boldsymbol{B}_0')_j +} \\ \overline{+\sum_{u=1}^{3}(\partial/\partial x_u')\Big\{\sum_{m,n=1}^{3}L'_{(uj),(mn)}(\boldsymbol{B}_0')\tfrac{1}{2}(\partial\Delta v_n'/\partial x_m' + \partial\Delta v_m'/\partial x_n')\Big\}\big]} = \\ = \overline{\Delta\tilde{g}_j'\big[-\varrho^{(0)}e^{(0)}c^{-1}(\Delta\boldsymbol{v}\wedge\boldsymbol{B}_0)_i + \sum_{u=1}^{3}(\partial/\partial x_u)\Big\{\sum_{m,n=1}^{3}L_{(ui),(mn)}(-\boldsymbol{B}_0)} \\ \overline{\tfrac{1}{2}(\partial\Delta v_n/\partial x_m + \partial\Delta v_m/\partial x_n)\Big\}\big]}. \qquad (i,\,j=1,2,3)$$

With (II.67) we find for (II.126)

$$\text{(II.127)}\qquad \sum_{u=1}^{3}(\partial/\partial x_u')\Big[\sum_{m,n=1}^{3}L'_{(ui),(mn)}(\boldsymbol{B}_0')\{\delta_{in}(\partial/\partial x_m')\,\delta(\boldsymbol{r}-\boldsymbol{r}') + \delta_{im}(\partial/\partial x_n')\delta(\boldsymbol{r}-\boldsymbol{r}')\}\Big] = \\ = \sum_{u=1}^{3}(\partial/\partial x_u)\Big[\sum_{m,n=1}^{3}L_{(ui),(mn)}(-\boldsymbol{B}_0)\{\delta_{jn}(\partial/\partial x_m)\delta(\boldsymbol{r}-\boldsymbol{r}') + \delta_{jm}(\partial/\partial x_n)\delta(\boldsymbol{r}-\boldsymbol{r}')\}\Big], \\ (i,\,j=1,2,3)$$

since a calculation shows that

$$\text{(II.128)}\qquad \overline{\Delta\tilde{g}_i\varrho^{(0)}e^{(0)}c^{-1}(\Delta\boldsymbol{v}'\wedge\boldsymbol{B}_0')_j} = -\overline{\Delta\tilde{g}_j'\varrho^{(0)}e^{(0)}c^{-1}(\Delta\boldsymbol{v}\wedge\boldsymbol{B}_0)_i}$$

is an identity, when (II.67) is applied. Eliminating δ-functions and Kronecker δ's we find from (II.127)

$$\text{(II.129)}\qquad L_{(uj),(mi)}(\boldsymbol{B}_0) + L_{(mj),(ui)}(\boldsymbol{B}_0) = L_{(ui),(mj)}(-\boldsymbol{B}_0) + L_{(mi),(uj)}(-\boldsymbol{B}_0)\,,$$

$$\text{(II.130)}\qquad \sum_{m=1}^{3}(\partial/\partial x_m)\,L_{(uj),(mi)}(\boldsymbol{B}_0) = \sum_{m=1}^{3}(\partial/\partial x_m)\,L_{(mi),(uj)}(-\boldsymbol{B}_0)\,,$$

with $i, j, m, u = 1, 2, 3$. With the usual boundary conditions (see § **8**), we get

$$\text{(II.131)}\qquad L_{(uj),(mi)}(\boldsymbol{B}_0) = L_{(mi),(uj)}(-\boldsymbol{B}_0)\,. \qquad (i,\,j,\,m,\,u=1,2,3)$$

These are 15 reciprocal relations amongst the 36 viscosity coefficients. This leaves us with 21 independent coefficients.

When (II.95)–(II.98) are inserted into the equations (II.75)–(II.77) and when (II.52)–(II.67) are applied, we find three identities for (II.75), (II.76) and (II.77). As there arise no special difficulties, we shall not give an explicit calculation here.

10. – Case of Non-Vanishing Average Barycentric Motion.

So far we have found only reciprocal relations (equations (II.123)–(II.125) and (II.131)) for the case, in which the (average) barycentric motion is zero. When, however, the centre of gravity has an (average) velocity different from zero, these reciprocal relations still hold, when $\boldsymbol{B}_0$ is replaced by $\boldsymbol{B}^*$, i.e. the magnetic induction in a coordinate frame moving with the centre of gravity (cf. remark at the end of § 7). So we find

$$\mathbf{L}_{ss}(\boldsymbol{B}^*) = \mathbf{L}^{\dagger}_{ss}(-\boldsymbol{B}^*)\,, \tag{II.132}$$

$$\mathbf{L}_{sk}(\boldsymbol{B}^*) = \mathbf{L}^{\dagger}_{ks}(-\boldsymbol{B}^*)\,, \qquad (k = 1, 2, \dots, n-1) \tag{II.133}$$

$$\mathbf{L}_{kl}(\boldsymbol{B}^*) = \mathbf{L}^{\dagger}_{lk}(-\boldsymbol{B}^*)\,, \qquad (k, l = 1, 2, \dots, n-1) \tag{II.134}$$

$$L_{(uj)(mi)}(\boldsymbol{B}^*) = L_{(mi)(uj)}(-\boldsymbol{B}^*)\,. \qquad (i\, j, m, u = 1, 2, 3) \tag{II.135}$$

These final results are the reciprocal relations for heat conduction (II.132), diffusion (II.134) and cross-effects (II.133) and also for viscosity (II.135) in mixtures of charged components with arbitrary movements in an electromagnetic field.

11. – Anisotropic Mixtures of Uncharged Components.

From the general case which has been treated in the foregoing one can easily derive the more particular cases in which the physical situation implies that not all the fluxes and forces are present.

The case of the anisotropic mixtures of n uncharged components, in which the phenomena of heat conduction, diffusion and cross-effects, and of viscous flow may occur, has been explicitly treated by DE GROOT and MAZUR [6] as a first example of their generalized Onsager theory. If we simply put $e_k = 0$, the phenomenological equations (II.38), (II.39) become

$$\boldsymbol{J}_s = -\,\mathbf{L}_{ss}\cdot\operatorname{grad} T - \sum_{l=1}^{n-1}\mathbf{L}_{sl}\cdot\operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n)\,, \tag{II.136}$$

$$\boldsymbol{J}_k = -\,\mathbf{L}_{ks}\cdot\operatorname{grad} T - \sum_{l=1}^{n-1}\mathbf{L}_{kl}\cdot\operatorname{grad}(\tilde{\mu}_l - \tilde{\mu}_n)\,, \qquad (k=1, 2, \dots, n-1) \tag{II.137}$$

where now $\tilde{\mu}_l = \mu_l + w_l$ (the system may be subject to external conservative forces). The equations for the viscous flow already coincide with the corres-

ponding equations which those authors have given. The reciprocal relations for the phenomenological coefficients, which have already been demonstrated in the previous sections, still hold.

12. – Galvanomagnetic and Thermomagnetic Phenomena in Anisotropic Metals.

Let us consider a system, consisting of a rigid ion lattice and of electrons, in an electromagnetic field and with a non uniform temperature. Equations (II.38), (II.39) reduce in this case, taking the barycentric velocity zero, to

$$\boldsymbol{J}_s = -\mathbf{L}_{ss}\cdot \operatorname{grad} T - \mathbf{L}_{se}\cdot\{\operatorname{grad}(\tilde{\mu}_e - \tilde{\mu}_i) - (e_e - e_i)c^{-1}\,\partial \boldsymbol{A}/\partial t\}\,, \tag{II.138}$$

$$\boldsymbol{J}_e = -\mathbf{L}_{es}\cdot \operatorname{grad} T - \mathbf{L}_{ee}\cdot\{\operatorname{grad}(\tilde{\mu}_e - \tilde{\mu}_i) - (e_e - e_i)c^{-1}\,\partial \boldsymbol{A}/\partial t\}\,; \tag{II.139}$$

moreover (II.36) reads

$$\boldsymbol{J}_i = -\boldsymbol{J}_e. \tag{II.140}$$

(index e indicating electrons, index i ions). By introducing the electric current and with (II.140) we obtain

$$\boldsymbol{i} = e_e \varrho_e \boldsymbol{v}_e + e_i \varrho_i \boldsymbol{v}_i = (e_e - e_i)\boldsymbol{J}_e\,. \tag{II.141}$$

Using now the approximation $e_e \gg e_i$ (the charge density per unit mass of electrons is much larger than the charge density per unit mass of the ions), we can finally write

$$\boldsymbol{J}_s = -\mathbf{L}_{ss}\cdot \operatorname{grad} T - \mathbf{L}_{se}\cdot\{\operatorname{grad}(\tilde{\mu}_e - \tilde{\mu}_i) - e_e c^{-1}\,\partial \boldsymbol{A}/\partial t\}\,, \tag{II.142}$$

$$\boldsymbol{i} = -e_e\mathbf{L}_{es}\cdot \operatorname{grad} T - e_e\mathbf{L}_{ee}\cdot\{\operatorname{grad}(\tilde{\mu}_e - \tilde{\mu}_i) - e_e c^{-1}\,\partial \boldsymbol{A}/\partial t\}\,. \tag{II.143}$$

Reciprocal relations (II.123)–(II.125) reduce to

$$\mathbf{L}_{ss}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{ss}(-\boldsymbol{B}_0), \qquad \mathbf{L}_{se}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{es}(-\boldsymbol{B}_0), \qquad \mathbf{L}_{ee}(\boldsymbol{B}_0) = \mathbf{L}^{\dagger}_{ee}(-\boldsymbol{B}_0)\,. \tag{II.144}$$

These equations enable us to treat the galvanomagnetic and thermomagnetic phenomena in anisotropic metals. They will be reduced in the next chapter to the case of isotropic metals, in which situation galvano- and thermomagnetic phenomena are actually studied.

Remark. These phenomena have been already treated, from the point of view of thermodynamics of irreversible processes, by CALLEN [12,13] and by MAZUR and PRIGOGINE [14]. Those authors, however, do not give a proof of the reciprocal relations: as it was usually done, they simply suppose that those relations are examples of the relations which ONSAGER [3] derived for scalar processes from the property of microscopic reversibility.

The same case of an anisotropic metal in a magnetic field has been explicitly studied [11] by introducing the physical simplifications from the beginning in the fundamental laws. We should like to say, however, that the reduction of the general case to a more particular case, and the introduction of the simplifications in the phenomenological equations, allow sometimes a more complete description of the phenomena.

The equations of this section, as it may be easily seen, include the case of electric conduction in anisotropic crystals in a magnetic field at uniform temperature, which has been explicitly treated by MAZUR and DE GROOT [10] by using their extension of Onsager's theory of reciprocal relations.

CHAPTER III.

EXPLICIT EXPRESSIONS FOR THE MEASURABLE EFFECTS IN ISOTROPIC METALS

1. – Introduction.

In the preceding chapter we have derived the Onsager relations for heat conduction, electrical conduction and cross-effects in anisotropic metals in the presence of a magnetic field. The purpose of this chapter is to derive the expressions for the observable coefficients usually defined for the case of isotropic metals [15].

The coefficients for the galvanomagnetic and thermomagnetic effects have already been given in local form [12-14,1], i.e. the various relations between electric field, temperature gradient, heat and electrical flow are expressed in terms of the phenomenological coefficients at a certain point of the system.

(12) H. B. CALLEN: *Phys. Rev.*, **73**, 1349 (1948).
(13) H. B. CALLEN: *Phys. Rev.*, **85**, 16 (1952).
(14) P. MAZUR and I. PRIGOGINE: *Journ. Phys. et Radium*, **12**, 616 (1951).
(15) R. FIESCHI, S. R. DE GROOT and P. MAZUR: *Physica*, **20**, 259 (1954).

However these local coefficients do not necessarily correspond to the experimentally measured coefficients ([16-18]), because « heterogeneous and homogeneous thermoelectric effects » and Peltier effects may also play a role, in addition to the purely galvano- and thermomagnetic effects. Consequently, of course, the relations among the various effects have also to be modified.

Besides the effects which have already been studied in local form by other authors ([12-14]), mentioned above, we have considered also the analogous effects in a longitudinal magnetic field, and the Peltier effect in a transversal and longitudinal magnetic field ([1,18,19]). In particular we remark that Thomson's second relation is still valid in the presence of a magnetic field.

2. – Phenomenological Equations.

Let us consider an isotropic metal in a magnetic field $\boldsymbol{B}$, parallel to the z-axis. As can easily be seen, the phenomenological equations (II.142), (II.143) for this case, neglecting the time derivative of the vector potential $\partial \boldsymbol{A}/\partial t$, (i.e. assuming that the electron motion is sufficiently slow to allow us to neglect the effect of an induced magnetic field), can be written as (*)

$$i_x = - L_{ee,xx} e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x - L_{ee,xy} e_e^{-1} \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial y - L_{es,xx} \partial T/\partial x - L_{es,xy} \partial T/\partial y, \tag{III.1}$$

$$i_y = L_{ee,xy} e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x - L_{ee,xx} e_e^{-1} \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial y + L_{es,xy} \partial T/\partial x - L_{es,xx} \partial T/\partial y, \tag{III.2}$$

$$i_z = - L_{ee,zz} e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial z - L_{es,zz}\, \partial T/\partial z\,, \tag{III.3}$$

$$J_{s,x} = - L_{se,xx} e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x - L_{se,xy} e_e^{-1} \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial y - L_{ss,xx} \partial T/\partial x - L_{ss,xy} \partial T/\partial y, \tag{III.4}$$

$$J_{s,y} = L_{se,xy} e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x - L_{se,xx} e_e^{-1} \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial y + L_{ss,xy} \partial T/\partial x - L_{ss,xx} \partial T/\partial y, \tag{III.5}$$

$$J_{s,z} = - L_{se,zz} e_e^{-1} \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial z - L_{ss,zz}\, \partial T/\partial z\,, \tag{III.6}$$

([16]) W. MEISSNER: *Handbuch der Experimental-Physik*, **11** (Leipzig, 1935).
([17]) W. GERLACH: *Handbuch der Physik*, **13**, Chapt. 6 (Berlin, 1928).
([18]) L. L. CAMPBELL: *Galvanomagnetic and thermomagnetic effects* (London, 1923).
([19]) G. LASKI: *Handbuch der Physik*, **13**, Chapt. 5 (Berlin, 1928).

(*) For the sake of symmetry the coefficients $\boldsymbol{L}_{es}$ and $\boldsymbol{L}_{se}$ of equations (II.142, 143) have been multiplied by e_e and the coefficients $\boldsymbol{L}_{ee}$ of equation (II.143) have been multiplied by e_e^2; these constant parameters have been included in the coefficients of equations (III.1-6).

The isotropy of the system requires, in addition to the following relations between the coefficients:

$$\text{(III.7)}\quad \begin{cases} L_{ee,xx}=L_{ee,yy}\,, & L_{es,xx}=L_{es,yy}\,, \quad L_{se,xx}=L_{se,yy}\,, \quad L_{ss,xx}=L_{ss,yy}\,, \\ L_{ee,xy}=-L_{ee,yx}\,, & L_{es,xy}=-L_{es,yx}\,, \quad L_{se,xy}=-L_{se,yx}\,, \quad L_{ss,xy}=-L_{ss,yx}\,, \end{cases}$$

which have already been taken into account in the equations (III.1)–(III.6), also that the coefficients $L_{ee,xx}$, $L_{es,xx}$, $L_{se,xx}$, $L_{ss,xx}$, $L_{ee,zz}$, $L_{es,zz}$, $L_{se,zz}$, and $L_{ss,zz}$ are even functions of the magnetic field, and that $L_{ee,xy}$, $L_{es,xy}$, $L_{se,xy}$ and $L_{ss,xy}$ are odd functions of the magnetic field.

Moreover the Onsager relations (II.144) hold between the coefficients:

$$\text{(III.8)}\quad L_{ee,ij}(\boldsymbol{B})=L_{ee,ji}(-\boldsymbol{B})\,,\; L_{es,ij}(\boldsymbol{B})=L_{se,ji}(-\boldsymbol{B})\,,\; L_{ss,ij}(\boldsymbol{B})=L_{ss,ji}(-\boldsymbol{B})\,,$$

where i and j denote Cartesian components. Some of these relations already follow from the isotropy of the system, the really new relations being

$$L_{es,xx}(\boldsymbol{B})=L_{se,xx}(-\boldsymbol{B})\,,$$

or, since $L_{se,xx}$ is an even function of $\boldsymbol{B}$,

$$\text{(III.9)}\quad L_{es,xx}(\boldsymbol{B})=L_{se,xx}(\boldsymbol{B})\,;$$

$$L_{es,xy}(\boldsymbol{B})=L_{se,yx}(-\boldsymbol{B})\,,$$

or, since $L_{se,xy}$ is an odd function of $\boldsymbol{B}$, and from (III. 7),

$$\text{(III.10)}\quad L_{es,xy}(\boldsymbol{B})=L_{se,xy}(\boldsymbol{B})\,;$$

$$L_{es,zz}(\boldsymbol{B})=L_{se,zz}(-\boldsymbol{B})\,,$$

or, since $L_{se,zz}$ is an even function of $\boldsymbol{B}$,

$$\text{(III.10}a\text{)}\quad L_{es,zz}(\boldsymbol{B})=L_{se,zz}(\boldsymbol{B})\,.$$

Remark. Other authors (12-14,1) have studied the case with fluxes and forces limited to the x-y coordinate plane. We have considered the more general case of fluxes and forces of arbitrary direction, in order to be able to deal also with the effects in a longitudinal magnetic field. The isotropy of the system requires that the coefficients with second sets of indices zx, zy, xz and yz vanish. So the equations (III.1), (III.2), (III.4) and (III.5) coincide with the equations which the quoted authors wrote down. In addition to these we have the equations (III.3) and (III.6), which are independent of the fluxes and forces in the x-y plane: they are the same equations which describe the thermoelectric effects, but the phenomenological coefficients are here of course functions of the magnetic field. These equations will enable us to treat

the effects in a longitudinal magnetic field. (In the anisotropic case the effects in a longitudinal field contain of course also cross-terms, pertaining to the perpendicular directions).

MAZUR and PRIGOGINE [11] pointed out that a different form of the phenomenological equations provides a more convenient way to deal with the problem in the case of a metal. Their form differs in two respects from the form used above and in [12]: 1st by choosing as independent variables the quantities which are directly controlled in the experiments (electric currents and temperature gradients), i.e. by changing the role of the electric flux and force, it is possible to obtain very simply the explicit expressions of the effects and the relations among them as functions of the phenomenological coefficients; 2nd moreover, a different choice of the fluxes and forces explained hereafter allows also the treatment of the effects in which the electric current does not vanish in the direction of the « adiabatical » insulation (*).

The new set of variables is obtained from the former set by a linear transformation which leaves the entropy production σ invariant. The entropy production (II.37) in our case is

$$\sigma = - T^{-1} \sum_{j=1}^{3} [J_{s,j}(\partial T/\partial x_j) - J_{e,j}(\partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x_j)]. \tag{III.11}$$

The phenomenological equations (III.1)–(III.6), where i_j and $J_{s,j}$ are the flows, $- e_e^{-1}\, \partial(\tilde{\mu}_e - \tilde{\mu}_i)/\partial x_j$ and $\partial T/\partial x_j$ the corresponding forces, are based on this expression. With the linear transformations

$$J_{\bar{q}} = T[J_s - (s_e - s_i)J_e]\,, \tag{III.12}$$

where $J_{\bar{q}}$ is called « reduced heat flow », s_e and s_i are the specific entropies of electrons and ions, (III.11) can be written as

$$\sigma = T^{-1} \sum_{j=1}^{3} (- J_{\bar{q},j} T^{-1}(\partial T/\partial x_j) + J_{e,j} \{- (e_e - e_i)\, \partial\varphi/\partial x_j - \\ - [\partial(\mu_e - \mu_i)/\partial x_j]_{T=\text{const}}\})\,, \tag{III.13}$$

or, since the total pressure and the concentrations of electrons and ions can be supposed uniform, $((\text{grad}\,\mu_e)_{T,P=\text{const}} = 0,\ (\text{grad}\,\mu_i)_{T,P=\text{const}} = 0)$ and with (II.141) and the approximation $e_e \gg e_i$,

$$\sigma = T^{-1} \sum_{j=1}^{3} \{- J_{\bar{q},j} T^{-1}(\partial T/\partial x_j) + i_j E_j\}\,, \tag{III.14}$$

where the electric field $E = -\,\text{grad}\,\varphi$ has been introduced.

It is now possible to write linear phenomenological relations which express

(*) In non-metallic systems the interchange of the role of electric flux and force is also preferable; but the different set of fluxes and forces is not suitable [13].

the fluxes i_i and $J_{\bar{q},i}$ as functions of the forces $-T^{-1}\partial T/\partial x_i$ and E_i. As it has been said, however, it is convenient to express the variables $J_{\bar{q},i}$ and E_i in terms of the other quantities.

$$\text{(III.15)} \qquad E_x = L_{ee,xx}i_x + L_{ee,xy}i_y - L_{es,xx}T^{-1}\,\partial T/\partial x - L_{es,xy}T^{-1}\,\partial T/\partial y\,,$$

$$\text{(III.16)} \qquad E_y = -L_{ee,xy}i_x + L_{ee,xx}i_y + L_{es,xy}T^{-1}\,\partial T/\partial x - L_{es,xx}T^{-1}\,\partial T/\partial y\,,$$

$$\text{(III.17)} \qquad E_z = L_{ee,zz}i_z - L_{es,zz}T^{-1}\,\partial T/\partial z\,,$$

$$\text{(III.18)} \qquad J_{\bar{q},x} = L_{se,xx}i_x + L_{se,xy}i_y - L_{ss,xx}T^{-1}\,\partial T/\partial x - L_{ss,xy}T^{-1}\,\partial T/\partial y\,,$$

$$\text{(III.19)} \qquad J_{\bar{q},y} = -L_{se,xy}i_x + L_{se,xx}i_y + L_{ss,xy}T^{-1}\,\partial T/\partial x - L_{ss,xx}T^{-1}\,\partial T/\partial y\,,$$

$$\text{(III.20)} \qquad J_{\bar{q},z} = L_{se,zz}i_z - L_{ss,zz}T^{-1}\,\partial T/\partial z\,.$$

Here the isotropy of the system has already been taken into account; from this condition it follows also that $L_{ee,xx}$, $L_{es,xx}$, $L_{se,xx}$, $L_{ss,xx}$, $L_{ee,zz}$, $L_{es,zz}$, $L_{se,zz}$ and $L_{ss,zz}$ are even functions of the magnetic field, while $L_{ee,xy}$, $L_{es,xy}$, $L_{se,xy}$ and $L_{ss,xy}$ are odd functions of the magnetic field. The validity of the Onsager relations is conserved by a linear transformation of fluxes and forces ([1]). Excluding the relations which are already given by the isotropy of the system, they read

$$\text{(III.21)} \qquad L_{es,xx}(\boldsymbol{B}) = -L_{se,xx}(-\boldsymbol{B})\ \{= -L_{se,xx}(\boldsymbol{B})\}\,,$$

$$\text{(III.22)} \qquad L_{es,xy}(\boldsymbol{B}) = -L_{se,yx}(-\boldsymbol{B})\ \{= -L_{se,xy}(\boldsymbol{B})\}\,,$$

$$\text{(III.23)} \qquad L_{es,zz}(\boldsymbol{B}) = -L_{se,zz}(-\boldsymbol{B})\ \{= -L_{se,zz}(\boldsymbol{B})\}\,,$$

where the minus sign results from the interchange between the electrical flux and force.

With the aid of these relations the independent phenomenological coefficients reduce to six for the equations of the components in the x-y plane, perpendicular to the magnetic field: $L_{ee,xx}$, $L_{ee,xy}$, $L_{es,xx}$, $L_{es,xy}$, $L_{ss,xx}$ and $L_{ss,xy}$, and to three for the equations of the components in the direction of the magnetic field: $L_{ee,zz}$, $L_{es,zz}$ and $L_{ss,zz}$.

3. – Local Effects and Relations among Them.

The effects we study are defined in MEISSNER ([16]) (see also GERLACH ([17])). The galvanomagnetic effects are caused by an electric current; the thermomagnetic effects are caused by a heat flow. In the transversal effects the primary current (heat or electric current) is perpendicular to the produced effect; in the longitudinal effects both variables have the same direction. An isothermal effect is defined in such a way that the temperature gradient in the direction perpendicular to the primary current is zero; for an adiabatic

effect no heat is allowed to flow in the direction perpendicular to the primary current.

Symmetry arguments on the reversal of the magnetic field suggest that the transversal effects are in first approximation proportional to B, the longitudinal effects to B^2. Several authors [12-18.1] therefore divide the coefficients of the various effects (see following table) by B or B^2 respectively, in order to obtain new coefficients, which are « independent » of B. We prefer, however, not to express explicitly the dependence of the coefficients on the magnetic field, since the experimental deviations from the foregoing simple dependence are too relevant. The even or odd parity of the effects with respect to the magnetic field is already contained in the even or odd parity of the phenomenological coefficients, as is shown in the preceding section.

Below we give a table with the definitions of the various local effects (14, with a transversal magnetic field, of the theoretically possible $8!\,2!\,3!\,3!=560$, and 4 with a longitudinal magnetic field), and their local expressions in terms of the phenomenological coefficients [12-11], which are easily derived from the equations (III.15)–(III.20).

When the isotropy of the system is taken into account, the 14 effects with the transversal field are expressed in terms of 9 independent phenomenological coefficients (cf. table); hence 5 relations exist, connecting the effects with each other, which are simply due to the symmetry of the system:

(III.24) $$R^l_a - R^l_i = - Q^t_i P^t\,,$$

(III.25) $$\lambda_a - \lambda_i = (S^t)^2 \lambda_i\,,$$

(III.26) $$R^t_a - R^t_i = Q^l_i P^t\,,$$

(III.27) $$Q^t_a - Q^t_i = Q^l_i S^t\,,$$

(III.26)–(III.27) (Heulinger relations)

(III.28) $$Q^l_a - Q^l_i = - Q^t_i S^t\,.$$

Moreover, the Onsager relations (III.21) and (III.22) reduce the number of the independent coefficients to 6; we then have among the effects 3 more relations:

(III.29) $$\lambda_i P^t = T Q^t_i\,, \qquad \text{(Bridgman relation)}$$

(III.30) $$\lambda_i P^l_i = T Q^l_i\,,$$

(III.31) $$\lambda_a P^l_a = T Q^l_a\,.$$

Owing to the Onsager relation (III.23), the 4 effects with the longitudinal field are expressed in terms of 3 independent phenomenological coefficients; hence one relation exists among them:

(III.32) $$\lambda^{lB} P^{lB} = T Q^{lB}\,.$$

TABLE I.

Galvanomagnetic effects

		Name of the effect	definition	conditions	effect
Transversal effects	I	Hall { isothermal: R_i^t	$\frac{E_y}{i_x}$	$i_y = 0,\ \frac{\partial T}{\partial x} = 0,\ \frac{\partial T}{\partial y} = 0.$	$-L_{ee,xy}$
	II	Hall { adiabatic: R_a^t	$\frac{E_y}{i_x}$	$i_y = 0,\ \frac{\partial T}{\partial x} = 0,\ J_{\bar{q},y} = 0.$	$-L_{ee,xy} + \frac{L_{es,xx}L_{se,xy}}{L_{ss,xx}}$
	III	Ettingshausen: P^t	$\frac{\partial T/\partial y}{i_x}$	$i_y = 0,\ \frac{\partial T}{\partial x} = 0,\ J_{\bar{q},y} = 0.$	$-T\frac{L_{se,xy}}{L_{ss,xx}}$
Longitudinal effects	VII	electric resistivity { isothermal: R_i^l	$\frac{E_x}{i_x}$	$i_y = 0,\ \frac{\partial T}{\partial x} = 0,\ \frac{\partial T}{\partial y} = 0.$	$L_{ee,xx}$
	VIII	electric resistivity { adiabatic: R_a^l	$\frac{E_x}{i_x}$	$i_y = 0,\ \frac{\partial T}{\partial x} = 0,\ J_{\bar{q},y} = 0.$	$L_{ee,xx} + \frac{L_{se,xy}L_{es,xy}}{L_{ss,xx}}$
in a transversal field	IX	Nernst { isothermal: P_i^l	$\frac{\partial T/\partial x}{i_x}$	$i_y = 0,\ J_{\bar{q},x} = 0,\ \frac{\partial T}{\partial y} = 0.$	$T\frac{L_{se,xx}}{L_{ss,xx}}$
	X	Nernst { adiabatic: P_a^l	$\frac{\partial T/\partial x}{i_x}$	$i_y = 0,\ J_{\bar{q},x} = 0,\ J_{\bar{q},y} = 0.$	$T\frac{L_{se,xx}L_{ss,xx} + L_{ss,xy}L_{se,xy}}{L_{ss,xx}^2 + L_{ss,xy}^2}$
Longitudinal effects	XV	electric resistivity: R^{lB}	$\frac{E_z}{i_z}$	$\frac{\partial T}{\partial z} = 0.$	$L_{ee,zz}$
in a longitudinal field	XVI	P^{lB}	$\frac{\partial T/\partial z}{i_z}$	$J_{\bar{q},z} = 0.$	$T\frac{L_{se,zz}}{L_{ss,zz}}$

TABLE II.

Thermomagnetic effects

		Name of the effect		definition	conditions	effect
Transversal effects	IV	Righi-Leduc: S^t		$\frac{\partial T/\partial y}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ J_{\bar{q},y} = 0.$	$\frac{L_{ss,xy}}{L_{ss,xx}}$
	V	Nernst	isothermal: Q_i^t	$\frac{E_y}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ \frac{\partial T}{\partial y} = 0.$	$T^{-1}L_{es,xy}$
	VI	Nernst	adiabatic: Q_a^t	$\frac{E_y}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ J_{\bar{q},y} = 0.$	$T^{-1}\left(L_{es,xy} - \frac{L_{es,xx}L_{ss,xy}}{L_{ss,xx}}\right)$
Longitudinal effects	XI	heat conductivity	isothermal: λ_i	$\frac{-J_{\bar{q},x}}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ \frac{\partial T}{\partial y} = 0.$	$T^{-1}L_{ss,xx}$
	XII	heat conductivity	adiabatic: λ_a	$\frac{-J_{\bar{q},x}}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ J_{\bar{q},y} = 0.$	$T^{-1}\left(L_{ss,xx} + \frac{L^2_{ss,xy}}{L_{ss,xx}}\right)$
in a transversal field	XIII	Ettingshausen-Nernst	isothermal: Q_i^l	$\frac{E_x}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ \frac{\partial T}{\partial y} = 0.$	$-T^{-1}L_{es\,xx}$
	XIV	Ettingshausen-Nernst	adiabatic: Q_a^l	$\frac{E_x}{\partial T/\partial x}$	$i_x = 0,\ i_y = 0,\ J_{\bar{q},y} = 0.$	$-T^{-1}\left(L_{es,xx} + \frac{L_{es,xy}L_{ss,xy}}{L_{ss,xx}}\right)$
Longitudinal effects	XVII	heat conductivity: λ^{lB}		$\frac{-J_{\bar{q},z}}{\partial T/\partial z}$	$i_z = 0.$	$T^{-1}L_{ss,zz}$
in a longitudinal field	XVIII	homogeneous thermo-electric effect: Q^{lB}		$\frac{E_z}{\partial T/\partial z}$	$i_z = 0.$	$-T^{-1}L_{es,zz}$

4. – Measurable Effects.

Let us consider a system consisting of a metallic sample A and of the metallic wires B, which connect two corresponding points on opposite faces of the sample with a condenser C (fig. 1). We call: $(\Delta\varphi)_i = (\varphi_1-\varphi_2)_i$ the potential difference which is measured (for instance) with a condenser connected with the corresponding points on the faces parallel to the plane $x_j x_k$ ($x_1 = x$, $x_2 = y$, $x_3 = z$; $i, j, k = 1, 2, 3$; $i \neq j \neq k$) (the figure illustrates the case $i \equiv y$); $(\Delta T)_i$ the temperature difference between the corresponding points on the faces parallel to the plane $x_j x_k (i \neq j \neq k)$; I_i the total electric current which is measured in the direction x_i; $I_{\bar{q},i}$ the total « reduced heat flow » which flows in the direction x_i. Moreover we suppose that i_i, $J_{\bar{q},i}$, $\partial\varphi/\partial x_i$ and $\partial T/\partial x_i$ are uniform inside the metal A. Then the following relations hold between the above defined measurable quantities and the corresponding « local » variables:

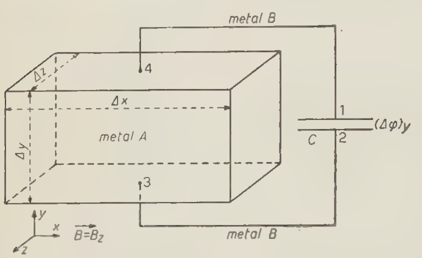

Fig. 1. – Sample with connecting wires.

$$(\text{III.33})\qquad (\varphi_4-\varphi_3)_i = \int\limits_{\Delta x_i} (\partial\varphi/\partial x_i)\, \mathrm{d}x_i = \Delta x_i\, (\partial\varphi/\partial x_i)\,,$$

$$(\text{III.34})\qquad (\Delta T)_i = \int\limits_{\Delta x_i} (\partial T/\partial x_i)\, \mathrm{d}x_i = \Delta x_i\, (\partial T/\partial x_i)\,,$$

$$(\text{III.35})\qquad I_i = \int\limits_{\Delta x_j}\int\limits_{\Delta x_k} i_i\, \mathrm{d}x_j\, \mathrm{d}x_k = \Delta x_j\, \Delta x_k i_i\,, \qquad (i \neq j \neq k)$$

$$(\text{III.36})\qquad I_{\bar{q},i} = \int\limits_{\Delta x_j}\int\limits_{\Delta x_k} J_{\bar{q},i}\, \mathrm{d}x_j\, \mathrm{d}x_k = \Delta x_j\, \Delta x_k J_{\bar{q},i}\,. \qquad (i \neq j \neq k)$$

In the experimental apparatus one has, besides the sample A, which is placed in the magnetic field $\boldsymbol{B}$, wires of a different metal B and junctions of the two different metals (Fig. 1: points 3 and 4). Consequently one should pay attention to the thermoelectric effects which may be superimposed on the pure galvano- and thermomagnetic effects.

When the temperature in the wires and at the junctions is different, a condenser records, in addition to the potential difference of galvano- or thermomagnetic origin, a thermoelectric potential difference which is given by the following two effects ([1]).

1) The « homogeneous thermoelectric effect », i.e. the potential difference

inside the single wires, is given by

$$\varphi_a - \varphi_b = - Q_B^{**}(T_a - T_b)/eT \,, \tag{III.37}$$

where the indices a and b indicate two points of a wire, Q^{**} is the « reduced heat of transfer » for the electrons in the metal B (i.e. the reduced heat transported by the unit of mass of the electrons in the state of uniform temperature, which can be expressed by the coefficients of the phenomenological equations for metal B) (*).

2) The « heterogeneous thermoelectric effect », i.e. the potential difference at the junction of two different metals A and B at the temperature T, is given by

$$(\varphi_A - \varphi_B)_T = e^{-1}(\mu_A - \mu_B)_T \,, \tag{III.38}$$

where μ_A and μ_B are the chemical potentials of electrons in the two metals (+).

The general equation for the resulting potential is obtained by adding up the homogeneous and heterogeneous potential differences in the circuit. One then obtains, for (III.33), (III.37) and (III.38), (Fig. 1),

$$\begin{aligned}(\Delta\varphi)_i &= (\varphi_1 - \varphi_2)_i = (\varphi_1 - \varphi_4)_B + (\varphi_B - \varphi_A)_{T_4} + (\varphi_4 - \varphi_3)_A + \\ &+ (\varphi_A - \varphi_B)_{T_3} + (\varphi_3 - \varphi_2)_B = e^{-1}T^{-1}Q_B^{**}(T_4 - T_1) + e^{-1}(\mu_A - \mu_B)T_4 + \\ &+ \int_{\Delta x_i} (\partial\varphi/\partial x_i)\, dx_i + e^{-1}(\mu_B - \mu_A)_{T_3} + e^{-1}T^{-1}Q_B^{**}(T_2 - T_3) = \\ &= e^{-1}T^{-1}Q_B^{**}(T_4 - T_1 + T_2 - T_3) + e^{-1}[\partial(\mu_A - \mu_B)/\partial T](T_4 - T_3) + \\ &+ (\partial\varphi/\partial x_i)\,\Delta x_i = e^{-1}T^{-1}Q_B^{**}(T_4 - T_1 + T_2 - T_3) + \\ &+ e^{-1}(s_B - s_A)(T_4 - T_3) + (\partial\varphi/\partial x_i)\,\Delta x_i \,.\end{aligned} \tag{III.39}$$

The quantities s_A and s_B are the specific entropies of electrons in the metals A and B, and in the last step of the derivation Gibbs' equation has been used.

(*) In the literature two sets of « heats of transfer » are used [20]: one of the sets consists of n quantities; each of these quantities may be interpreted as the heat transported per unit flow of matter of a certain chemical component. The other sets consists of $n-1$ quantities, where each quantity may be interpreted as the heat transported per unit flow of matter of a certain chemical component relative to the barycentric motion. Since in our case $c_e \ll c_i$ (c_e is the concentration of electrons, c_i the concentration of ions), it is easily seen [20] that the « heats of transfer » of the latter set coincide with the first $n-1$ heats of transfer of the first set mentioned (in our case $n=2$). Moreover, since relative and absolute flows coincide, the corresponding two kinds of « reduced heat of transfer » also coincide, and no further specification is necessary.

(+) Here and in the following we let $e_e = e$.

(20) G. A. Kluitenberg, S. R. de Groot and P. Mazur: *Physica*, 19, 1079 (1953), § 6.

If the temperature gradient vanishes in the direction in which the potential difference is measured, $(T_1 = T_2 = T_3 = T_4)$, the « homogeneous thermoelectric effect » also vanishes, the « heterogeneous effects » at the two junctions are equal and with opposite sign, and one has simply

$$(\Delta\varphi)_i = (\partial\varphi/\partial x_i)\,\Delta x_i = -E_i\,\Delta x_i\,. \tag{III.40}$$

The isothermal Hall (I) and Nernst (V) effects, and the isothermal (VII) and adiabatic (VIII) electrical resistivity belong to this case.

For the adiabatic Hall (II) and Nernst (VI) effects, one has $T_3 \neq T_4$, but $T_1 = T_4$ and $T_2 = T_3$ (from the adiabatical insulation condition for the y direction). The homogeneous thermoelectric effect then vanishes and (III.39) becomes

$$(\Delta\varphi)_i = e^{-1}(s_B - s_A)(T_4 - T_3) - E_i\,\Delta x_i\,. \tag{III.41}$$

For the isothermal (XIII) and adiabatic (XIV) Ettingshausen-Nernst effects the homogeneous thermoelectric effect can also be present, since the adiabatical insulation in the direction in which the electric field is measured is no longer necessary; (III.41) is then again valid only for the case that the temperature gradient is zero in the wires which connect the metal A with the condenser. In the following we shall consider this case in order to have consistency in the definitions and more simplicity in the relations among the various effects. In § **6**, however, to treat the Peltier effect, it is convenient to consider the more general case in which a temperature gradient in the wires is also present.

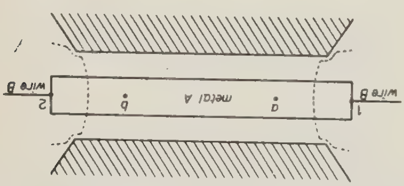

Fig. 2. - Sample in magnetic field. (Point 3 is at the left-hand dotted line in the metal; point 4 is at the right-hand dotted line in the metal).

If sample and wires are made of the same metal, the heterogeneous thermoelectric effect is the potential difference at the « junctions » between the metal inside the magnetic field and the metal outside: s_A is then the specific entropy of electrons for the metal inside the magnetic field, while s_B is the specific entropy of electrons in the metal without magnetic field (see the end of § **6**); T_3 and T_4 are the temperature at the points where the magnetic field drops to zero.

If the boundary of the magnetic field does not coincide with the junctions between the two metals, but part of metal A is left outside the field, one has 4 discontinuity points for the electric potential (fig. 2: points 1, 2, 3 and 4), and homogeneous thermoelectric effects both in the wires B and in the parts

of A outside the field (fig. 2: the dotted lines indicate the boundaries of the magnetic field).

It is now easy to give the relations between measurable and local effects. The coefficients with an upper bar indicate measurable coefficients, while without the bar they indicate local coefficients.

The isothermal Hall coefficient (I) is defined by $\bar{R}_i^t = -(\Delta\varphi)_y \Delta z / I_x$, when $I_y = 0$, $(\Delta T)_x = 0$ and $(\Delta T)_y = 0$. From (III.35) and (III.40) one has: $\bar{R}_i^t = E_y \Delta y \Delta z / i_x \Delta y \Delta z = R_i^t$. In a similar way it is immediately seen that the following observable effects coincide with the local effects. (For the sake of brevity we shall not write the conditions which define every observable effect, as they coincide with the conditions for the corresponding local effect). Isothermal Nernst coefficient (V): $\bar{Q}_i^t = -(\Delta\varphi)_y \Delta x / (\Delta T)_x \Delta y = Q_i^t$. (As a matter of fact temperature and potential differences in the x (longitudinal) direction are not measured at the ends of the sample A. In these cases, hence Δx is the distance between the points where the measurements are performed (fig. 2, points a and b)). Ettingshausen coefficient (III): $\bar{P}^t = (\Delta T)_y \Delta z / I_x = P^t$. Righi-Leduc coefficient (IV): $\bar{S}^t = (\Delta T_y) \Delta x / (\Delta T)_x \Delta y = S^t$. Isothermal (VII) and adiabatic (VIII) electric resistivity: $\bar{R}^l = -(\Delta\varphi)_x \Delta y \Delta z / I_x \Delta x$; $\bar{R}_i^l = R_i^l$, $\bar{R}_a^l = R_a^l$. Isothermal (XI) and adiabatic (XII) heat conductivity: $\bar{\lambda} = -I_{\bar{q},x} \Delta x / (\Delta T)_x \Delta y \Delta z$; $\bar{\lambda}_i = \lambda_i$, $\bar{\lambda}_a = \lambda_a$.

For the adiabatic Hall effect (II), as we have said in the foregoing, the influence of the potential difference at one junction is no longer compensated by the potential difference at the other junction, since the temperature at the two junctions is not the same. So we have, from (III.41),

$$\bar{R}_a^t = -(\Delta\varphi)_y \Delta z / I_x = (\Delta z / i_x \Delta y \Delta z)[E_y \Delta y + e^{-1}(s_B - s_A)\{-(\Delta T)_y\}] =$$
$$= R_a^t + e^{-1}(s_A - s_B)\{(\Delta T)_y / i_x \Delta y\},$$

or, for the definition of effect (III) in the table,

$$\bar{R}_a^t = R_a^t + e^{-1}(s_A - s_B)P^t \,. \tag{III.42}$$

In an analogous way one finds, for the adiabatic Nernst coefficient (VI),

$$\bar{Q}_a^t = -(\Delta\varphi)_y \Delta x / (\Delta T)_x \Delta y = Q_a^t + e^{-1}(s_A - s_B)S^t \,, \tag{III.43}$$

where the definition of effect (IV) in the table has been used. This correction for the measurable adiabatic Nernst coefficient has been given already by Campbell ([18]).

For the longitudinal isothermal Ettingshausen-Nernst coefficient (XIII) one has

$$\bar{Q}_i^l = -(\Delta\varphi)_x / (\Delta T)_x = Q_i^l + e^{-1}(s_A - s_B) \,, \tag{III.44}$$

and for the longitudinal adiabatic Ettingshausen-Nernst coefficient (XIV)

$$\bar{Q}^l_a = Q^l_a + e^{-1}(s_A - s_B)\,. \tag{III.45}$$

As we have remarked, these last two results are valid for the case in which the temperature gradient is zero in the wires. If the temperature gradient is non-vanishing and uniform (and so also $T_1 = T_2$), one has from (III.31)

$$\bar{Q}^l_i = Q^l_i + \{e^{-1}(s_A - s_B) - e^{-1}T^{-1}Q^{**}_B\}\,, \tag{III.46}$$

and

$$\bar{Q}^l_a = Q^l_a + \{e^{-1}(s_A - s_B) - e^{-1}T^{-1}Q^{**}_B\}\,. \tag{III.47}$$

The longitudinal isothermal (IX) and adiabatic (X) Nernst effects are the only two effects in which the electric current in the direction, in which one measures the temperature difference, does not vanish. One has therefore an evolution of heat at the junctions, which modifies the temperature difference due to the pure galvanomagnetic effect. However, the effect of heating at the junction should be corrected with suitable experimental arrangement, as it would be complicated to consider explicitly its influence on the expressions of the two galvanomagnetic coefficients. Apart from that, the experimental coefficients coincide with the local ones. The conditions supposed in this case are hardly to be obtained practically, as we must assume the possibility of the existence of an electric current without an accompanying «reduced heat flow». Nevertheless, the case is of theoretical interest. Longitudinal isothermal and adiabatic Nernst coefficients:

$$\bar{P}^l \equiv (\Delta T)_x \Delta y\, \Delta z / I_x \Delta x\,; \quad \bar{P}^l_i = P^l_i\,, \quad \bar{P}^l_a = P^l_a\,.$$

For the longitudinal effects in a longitudinal magnetic field again the same considerations hold as for the corresponding longitudinal effects in a transversal field. Two of them (XV) and (XVII) are unchanged: $\bar{R}^{lB} = R^{lB}$ and $\bar{\lambda}^{lB} = \lambda^{lB}$. The thermoelectric power (XVIII) $\bar{Q}^{lB} = -(\Delta\varphi)_z/(\Delta T)_z$ is given by

$$\bar{Q}^{lB} = Q^{lB} + e^{-1}(s_A - s_B)\,, \tag{III.48}$$

if the temperature gradient is zero in the wires, and by

$$\bar{Q}^{lB} = Q^{lB} + \{e^{-1}(s_A - s_B) - e^{-1}T^{-1}Q^{**}_B\}\,, \tag{III.49}$$

if the temperature gradient is non-vanishing and uniform. The coefficient $\bar{P}^{lB}$ (XVI) is again affected by the influence of an evolution of heat at the junctions, which cannot be expressed explicitly.

5. – Relations Among the Measurable Coefficients.

We have seen that the local expressions of the four effects II, VI, XIII and XIV, where the temperature gradient does not vanish in the direction in which the electric field is measured, must be modified, because for these cases a thermoelectric potential difference is superimposed on the galvano- or thermomagnetic potential difference. Consequently, some of the relations among the various effects are also modified.

The relations (III.24), (III.27) and (III.28) remain unmodified because the local and the measurable coefficients, which appear in them, coincide.

It is easy to see that the relations (III.29), (III.30) and (III.31), expressed in the measurable coefficients, remain unchanged, although the expressions for the measurable coefficients are different from the expressions for the local coefficients.

Relations (III.25) and (III.26) are no longer valid when the measurable coefficients are substituted for the local coefficients. The relations then become

$$\bar{\lambda}_i \bar{P}_i^l = T\bar{Q}_i^l + e^{-1}(s_A - s_B) \tag{III.50}$$

and

$$\bar{\lambda}_a \bar{P}_a^l = T\bar{Q}_a^l + e^{-1}(s_A - s_B)\,, \tag{III.51}$$

or, by eliminating the specific entropies,

$$\bar{\lambda}_i \bar{P}_i^l - \bar{\lambda}_a \bar{P}_a^l = T(\bar{Q}_i^l - \bar{Q}_a^l)\,. \tag{III.52}$$

Also the relation (III.32) for the longitudinal effects in a longitudinal field is not valid any more for the measurable effects. Instead of it we have

$$\bar{\lambda}^{lB} \bar{P}^{lB} = T\bar{Q}^{lB} + e^{-1}(s_A - s_B)\,. \tag{III.53}$$

In this way finally all relations among measurable effects have been found.

6. – Peltier Effect in a Magnetic Field and Thomson's second Relation.

In this section we wish to extend the theory of thermoelectricity [1] to the case in which a magnetic field is present.

In order to study the Peltier effect in a magnetic field it is convenient to extend the definition of « reduced heat of transfer ». In a transversal magnetic field we define an « isothermal reduced heat of transfer » $(Q^{**})_i^{tB} = e^{-1}(J_{\bar{q},x}/i_x)_i^{tB}$,

with the conditions $i_y = 0$, $\partial T/\partial x = 0$, $\partial T/\partial y = 0$; and an « adiabatical reduced heat of transfer », $(Q^{**})_a^{tB} \cdot e^{-1}(J_{\bar{q},x}/i_x)_a^{tB}$, with the conditions $i_y = 0$, $\partial T/\partial x = 0$, $J_{\bar{q},y} = 0$. From the equations (III.18) and (III.19) we have then $(Q^{**})_i^{tB} = e^{-1}L_{se,xx}$, or, with the Onsager relation (III.21),

$$(Q^{**})_i^{tB} = - e^{-1}L_{es,xx}\,; \tag{III.54}$$

$(Q^{**})_a^{tB} = e^{-1}(L_{se,xx} + L_{ss,xy}L_{se,xy}/L_{ss,xx})$, or, with the Onsager relations (III.21) and (III. 22),

$$(Q^{**})_a^{tB} = e^{-1}(- L_{es,xx} - L_{ss,xy}L_{es,xy}/L_{ss,xx})\,. \tag{III.55}$$

In a longitudinal magnetic field we define the reduced heat of transfer $(Q^{**})^{lB} = e^{-1}(J_{\bar{q},z}/i_z)^{lB}$, with the condition $\partial T/\partial z = 0$. From equation (III.20) we have

$(Q^{**})^{lB} = e^{-1}L_{se,zz}$, or, with the Onsager relation (III.23),

$$(Q^{**})^{lB} = - e^{-1}L_{es,zz}\,. \tag{III.56}$$

(The coefficients here defined are, of course, functions of the magnetic field).

One easily sees that the following local relations hold:

$$e^{-1}(Q^{**})_i^{tB} = TQ_i^l\,, \quad e^{-1}(Q^{**})_a^{tB} = TQ_a^l\,, \quad e^{-1}(Q^{**})^{lB} = TQ^{lB}\,. \tag{III.57}$$

These relations are, in a certain way, the « local » analoga of the « measurable » second relation of Thomson.

The Peltier coefficient is defined as the absorbed heat per unit flow of an electrical current across an isothermal junction from metal A to metal B. We can define two coefficients for the case of a transversal magnetic field: an isothermal Peltier coefficient: $(\Pi_{AB})_i^{tB}$, with the conditions $i_y = 0$, $\partial T/\partial x = 0$, $\partial T/\partial y = 0$: an adiabatic Peltier coefficient: $(\Pi_{AB})_a^{tB}$, with the conditions $i_y = 0$, $\partial T/\partial x = 0$, $J_{\bar{q},y} = 0$; and a coefficient for the case of a longitudinal magnetic field: $(\Pi_{AB})^{lB}$, with the condition $\partial T/\partial z = 0$.

The evolution of heat at the junction between a metal A and a metal B can be found (1) as the difference between the entropy flows before and after the junction, multiplied by the temperature T, since there is no entropy source at the surface separating the two metals. For the isothermal and adiabatic case in metal A inside the transversal magnetic field we have, from (III.12), (III.18) and (III.19),

$$T(J_{s,x})_i = (J_{\bar{q},x})_i + e^{-1}Ts_A i_x = (L_{se,xx} + e^{-1}Ts_A)i_x = e^{-1}\{(Q_A^{**})_i^{tB} + Ts_A\}i_x\,, \tag{III.58}$$

$$\begin{aligned} T(J_{s,x})_a &= (J_{\bar{q},x})_a + e^{-1}Ts_A i_x = (L_{se,xx} + L_{ss,xy}L_{se,xy}/L_{ss,xx} + e^{-1}Ts_A)i_x = \\ &= e^{-1}\{(Q_A^{**})_a^{tB} + Ts_A\}i_x\,. \end{aligned} \tag{III.59}$$

From (III.12) and (III.20), for metal A inside a longitudinal magnetic field, we have

$$\text{(III.60)}\qquad TJ_{s,z} = J_{\tilde{q},z} + e^{-1}Ts_A i_z = (L_{se,zz} + e^{-1}Ts_A)i_z = e^{-1}\{(Q_A^{**})^{lB} + Ts_A\}i_z\,.$$

In the wire B we have (1)

$$\text{(III.61)}\qquad TJ_{s,j} = e^{-1}(Q_B^{**} + Ts_B)i_j\,. \qquad (j = 1, 2, 3)$$

The difference between (III.61) ($i = 1$) and (III.58), divided by the electric current i_x gives

$$\text{(III.62)}\qquad (\Pi_{AB})_i^{tB} = e^{-1}\{Q_B^{**} - (Q_A^{**})_i^{tB} + T(s_B - s_A)\}\,.$$

The difference between (III.61) ($i = 1$) and (III.59), divided by the electric current i_x gives

$$\text{(III.63)}\qquad (\Pi_{AB})_a^{tB} = e^{-1}\{Q_B^{**} - (Q_A^{**})_a^{tB} + T(s_B - s_A)\}\,.$$

The difference between (III.61) ($i = 3$) and (III.60), divided by the electric current i_z gives

$$\text{(III.64)}\qquad (\Pi_{AB})^{lB} = e^{-1}\{Q_B^{**} - (Q_A^{**})^{lB} + T(s_B - s_A)\}\,.$$

From (III.46), (III.57) and (III.62) one has

$$\text{(III.65)}\qquad (\Pi_{AB})_i^{tB} = -T\bar{Q}_i^l\,;$$

from (III.47), (III.57) and (III.63) one has

$$\text{(III.66)}\qquad (\Pi_{AB})_a^{tB} = -T\bar{Q}_a^l\,;$$

from (III.49), (III.57) and (III.64) one has

$$\text{(III.67)}\qquad (\Pi_{AB})^{lB} = -T\bar{Q}^{lB}\,.$$

The coefficients $\bar{Q}_i^l$, $\bar{Q}_a^l$ and $\bar{Q}^{lB}$ can properly be called « *thermoelectric power in a* (transversal or longitudinal) *magnetic field* »; hence it is correct to say that the relations (III.65), (III.66) and (III.67) are extensions of Thomson's second relation, to the system consisting of a metal subject to a magnetic field and a metal outside the magnetic field.

Remark. We should like to stress again the fact that, when the whole system is made of the same metal, the effects between the metal inside and outside the magnetic field can be studied. In such a case s_A indicates the specific entropy of electrons for the metal inside the field, (when the metal is isotropic it does not matter whether the field is transversal or longitudinal) and s_B indicates the specific entropy of electrons for the metal outside the magnetic field. As a matter of fact Onsager relations have been demonstrated, in chapter II, for a metal without magnetization or polarization. However, they can easily be extended to the present case, in which the influence of the induced field is negligible. As MAZUR and PRIGOGINE (8) have shown, the local entropy production does not change, formally, when magnetization is also taken into account: the chemical potential μ only must be corrected with an additive term depending on the magnetic susceptibility of the metal and on the magnetic field. Since in our case both quantities are constant, the Onsager relations can be derived in the way outlined at the end of (11).

Appendix on Tensor Notation.

We have used the Milne system of tensor notations (21) in this paper. The exterior product of an ordered pair of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is a tensor

$$\mathbf{T} = \boldsymbol{a}\,\boldsymbol{b}, \quad \text{components:} \quad T_{ik} = a_i b_k\,.$$

The divergence of a tensor and the gradient of a vector are written as

$$\operatorname{Div}\mathbf{T}, \quad \text{components:} \quad (\operatorname{Div}\mathbf{T})_i = \textstyle\sum_k (\partial/\partial x_k) T_{ki}\,,$$

$$\operatorname{Grad}\boldsymbol{a}, \quad \text{components:} \quad (\operatorname{Grad}\boldsymbol{a})_{ik} = (\partial/\partial x_i) a_k\,.$$

The interior product between a tensor and a vector is denoted by

$$\mathbf{T}\cdot\boldsymbol{a}, \quad \text{components:} \quad (\mathbf{T}\cdot\boldsymbol{a})_i = \textstyle\sum_k T_{ik} a_k\,,$$

$$\boldsymbol{a}\cdot\mathbf{T}, \quad \text{components:} \quad (\boldsymbol{a}\cdot\mathbf{T})_i = \textstyle\sum_k a_k T_{ki}\,.$$

Finally

$$\mathbf{T}:\mathbf{U} = \textstyle\sum_{i,k} T_{ik} U_{ki}$$

is the interior product of two tensors $\mathbf{T}$ and $\mathbf{U}$ contracted twice.

(21) L. ROSENFELD: *Theory of Electrons* (Amsterdam, 1951).

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A Causal Interpretation of the Pauli Equation (A).

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Introduction.

In various previous papers ([1-8]), several new interpretations of the non-relativistic quantum theory without spin have been proposed. Although these new interpretations differ in various details, they all have in common that they explain the quantum theory in terms of continuous and causally determined motions of various kinds of entities, such as fields and bodies, which are assumed to exist objectively at the microscopic level. Thus far, those interpretations which have been carried far enough to demonstrate in full

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detail their ability to explain causally all of the features of the Schrödinger equation without spin have followed one of two general lines of approach.

The first of these general lines, initiated originally by DE BROGLIE (1), and later carried to its logical conclusions by one of the authors of the present paper (3), involves the notion that the Schrödinger wave function, $\Psi(\boldsymbol{x}, t)$, represents an objectively real but qualitatively new kind of field of force that influences the motion of a body, which latter has a well defined location, $\xi(t)$, varying continuously and in a causally determined way with the passage of time. This new field of force produces no important effects at the macroscopic level, but at the atomic level it is, as has been shown in various papers referred to above, able to explain the characteristic new quantum-mechanical properties of matter, which manifest themselves strongly only at this level.

The second general type of causal explanation of the quantum theory is along the lines of the hydrodynamic model; proposed originally by MADELUNG (2), and later extended by TAKABAYASI (4) and by SCHÖNBERG (7). In this model, $|\Psi|^2$ represents the density of a fluid, while $\nabla S/m$ represents its local stream velocity (where $\Psi = R\exp[iS/\hbar]$). In another paper (8), this model has been completed with the aid of the postulate that there is a stable particle-like inhomogeneity in the fluid that moves with the local stream velocity. This inhomogeneity plays a role analogous to that played by the body in the interpretations initiated by DE BROGLIE, while the fluid plays a role analogous to that played by the Ψ field.

In the present paper, we shall find it convenient to work in terms of the hydrodynamic model, which we shall extend with the aid of various new postulates to the Pauli equation describing an electron with spin. We shall therefore now summarize a few important aspects of the hydrodynamic model without spin, in order to facilitate the description of the nex features that are needed for the treatment of spin. We first write Schrödinger's equation in the well-known form

$$\frac{\partial R^2}{\partial t} + \operatorname{div}\left(R^2\frac{\nabla S}{m}\right) = 0\,, \tag{1}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\,\frac{\nabla^2 R}{R} = 0\,. \tag{2}$$

In terms of the Madelung interpretation, equation (1) then represents the conservation of fluid, while (2) represents the equation which determines the velocity potential, S, in terms of the classical potential, V, and the « quantum potential » $U = -(\hbar^2/2m)\nabla^2 R/R = (\hbar^2/4m)((\nabla\varrho)^2/2\varrho^2 - \nabla^2\varrho/\varrho)$. As shown by TAKABAYASI (4) and by SCHÖNBERG (7), the quantum potential can be interpreted in terms of a kind of internal stress in the fluid, which depends not,

however, on ϱ itself as is the case with pressures and other internal stresses in an ordinary macroscopic fluid, but rather, on derivatives of ϱ.

In one of the papers referred to previously ([8]), the additional assumption was made that the fluid is in a state of irregular fluctuation resembling turbulent motion, in which the actual density ϱ, and the actual velocity $\boldsymbol{v}$, fluctuate more or less at random around $|\Psi|^2$ and $\nabla S/m$ respectively as means. In these fluctuations, it is permissible to assume that $\boldsymbol{v}$ has vortex motions, which however average out to zero. As a result of these fluctuations, elements of fluid on any one of the mean lines of flow associated with the Madelung fluid are continually moving in an irregular way to other lines of flow, and this process tends to produce a more or less random « mixing » of the fluid. Like one of the fluid elements, the body-like inhomogeneity, which is carried along by the fluid, follows a very irregular path. It is then quite easily shown that a statistical ensemble of such systems with an arbitrary probability density, $P(\boldsymbol{x})$, of inhomogeneities eventually decays into one with $P=|\Psi|^2$. Thus, the irregular fluctuations in the motions of the Madelung fluid provide a model explaining in a natural way a possible physical origin for the statistical distributions of the quantum theory ([9]).

2. – Introduction of "Intrinsic Spin" in Hydrodynamical Model.

In the present paper and in a subsequent paper, we shall extend the hydrodynamic model to a treatment of the Pauli equation for a spinning electron. Now, the Pauli equation is ([10])

$$i\hbar\frac{\partial\Psi_a}{\partial t}=-\frac{\hbar^2}{2m}\left(\nabla-ie\frac{\boldsymbol{A}}{c}\right)^2\Psi_a+V\Psi_a+\frac{e\hbar}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{\mathfrak{H}})\Psi_a\,, \tag{3}$$

where Ψ_b represents a two component spinor with index, a, while $\boldsymbol{A}$ is the electromagnetic vector potential, V, the total external potential energy (electromagnetic and otherwise), and $\boldsymbol{\mathfrak{H}}$ the magnetic field.

As in the case of the Madelung model of Schrödinger's equation, we must

([9]) In a previous paper (see third paper of reference ([3])) a similar theorem was proved for the model in which the Ψ function is assumed to represent a field of force. In this case, the random fluctuations come from perturbations arising outside the system under investigation.

([10]) We do not include the spin orbit coupling term, $\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{\epsilon}$, because a consistent treatment of this term requires a relativistic theory, since it is of the same order of magnitude as the Thomas precession. This term must be dealt with in terms of a causal interpretation of the Dirac equation.

define a fluid density and a fluid velocity. Now, it is well known that the Pauli equation admits a conserved charge and current given (except for a factor of e) by

$$\varrho = \Psi^*\Psi\,, \tag{4-a}$$

$$\boldsymbol{j} = \frac{h}{2mi}\,(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{e}{c}\,\boldsymbol{A}\,. \tag{4-b}$$

Writing $\boldsymbol{j} = \varrho\boldsymbol{v}$, we may then define the velocity as

$$\boldsymbol{v} = \frac{h}{2mi}\,\frac{(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)}{\Psi^*\Psi} - \frac{e}{c}\,\boldsymbol{A}\,. \tag{4-c}$$

It is then evidently consistent to assume that the fluid density is given by $\varrho = \Psi^*\Psi$, since the conservation equation obtained from the Pauli equation can then be interpreted as describing the conservation of fluid.

ϱ and $\boldsymbol{v}$ do not, however, permit a complete definition of the physical meaning of the Ψ function; for since Ψ is a spinor, it contains four independent quantities, as indicated below

$$\Psi = \begin{pmatrix} a + ib \\ c + id \end{pmatrix}, \tag{5}$$

where the a, b, c, d are *real*. On the other hand (4-a), defines only $\varrho = a^2b^2 + c^2 + d^2$ while (4-b) defines only the derivatives of the a, b, c, d.

In order to define all parts of Ψ more directly in terms of physical properties of the fluid, we shall therefore have to assume some new properties for the fluid. Now, since the Pauli equation deals with the electron spin, it seems natural to assume that our fluid should have a new property, which we may call an *intrinsic angular momentum* connected with the spin. By this, we mean that the total angular momentum density of the fluid should include, in addition to the « orbital » contribution of $m\varrho(\boldsymbol{r}\times\boldsymbol{v})$, an additional contribution depending on some parameters connected with the internal motions of each fluid element. To obtain a possible physical picture of where such an intrinsic angular momentum could come from, we may suppose, for the sake of discussion, that the fluid is constituted of molecules, and that the spin motion of the constituent molecules may contribute to the total angular momentum of the system. On the other hand, the same result could be achieved in a much more general way. For if the fluid has an inhomogeneous structure, then these inhomogeneities will in general have a certain inertia. If a fluid element is turning, the inhomogeneities will turn with it and contribute to

the total angular momentum. Such inhomogeneities might, for example, be stable or semi-stable pulse-like structures formed in the fluid itself, or they might be small highly localized vortices or eddies. As far as our purposes in this paper are concerned, however, the origin of the intrinsic angular momentum of the fluid is irrelevant. All that is relevant is the assumption that, for one reason or another, such an intrinsic angular momentum exists. In order to have a convenient model in terms of which we can work, however, we shall assume in this paper that the intrinsic angular momentum is due to the turning of very small quasi-rigid bodies of which the fluid is supposed to be constituted (just as ordinary macroscopic fluids are constituted of molecules). We shall see that this simplified model is adequate for giving a causal explanation of the Pauli equation. On the other hand, there is no reason inherent in the model why the bodies must always be rigid; or, indeed, even why the inhomogeneities must necessarily be regarded as arising in distinct bodies out of which the fluid is supposed to be constituted. Nevertheless, under the conditions in which the Pauli equation applies, we shall assume that the fluid acts so nearly as if it were composed of distinct rigid spinning bodies, that we may use this model as a simplifying abstraction as one, for example frequently simplifies the treatment of atoms by replacing them by idealized mass points even though they really have finite sizes.

3. – Kinematic Description of Rotations.

We shall now proceed to develop a kinematic description of the rotations of a body, which is particularly easy to apply to the Pauli equation, because it works in terms of spinors of the same kind as appear in this equation.

The first problem is to specify the state of rotation of a body. This can be done in terms of the three Euler angles, θ, φ and ψ, where θ represents the angle of the principal axis (1) with a Z axis fixed in space, φ represents the angle that the projection of this axis makes on the X-Y plane, and ψ represents the angle of rotation about the principal axis (1) relative to the intersections of the plane of the principal axes (2) and (3), with the X-Y plane.

We wish now, however, to connect these angles to a spinor. To do this, let us imagine that we always start with the body in a standard orientation, in which its principal axis (1) is directed along the Z axis, which we have fixed in space. We then make a rotation, $R(\theta, \varphi, \psi)$, which carries the body from its standard orientation to its actual orientation. This rotation can be carried out in three steps. First, we make a rotation, $R_1(\psi)$ through an angle ψ about the Z axis fixed in space (which is at this time also the principal axis (1) of the body). We then make a rotation $R_2(\theta)$ about the X axis, of an angle, θ. Then we make a rotation $R_3(\varphi)$ again about the Z axis fixed in space (but this

time the principal axis of the body is no longer parallel to the Z axis). If the reader will draw a diagram, he will readilly convince himself that these rotations give just those described by the Euler angles [11].

Let us now carry out these operations mathematically. We shall assume that the state of the body at rest is to be represented by what we shall call the standard unit spinor [12] $\beta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. When we carry out the rotations described above, we shall see that a rotation involving arbitrary angles, θ, φ, ψ, can lead to an arbitrary unit spinor (such that $\beta^*\beta = 1$). Thus, any unit spinor can be interpreted as a specification on the angles of rotation of a body. In terms of spinor notation, the first rotation $R_1(\psi)$ is represented by the matrix

$$\exp\left[i\sigma_z\psi/2\right] = \cos\psi/2 + i\sigma_z \sin\psi/2 \,.$$

Applying this to the standard unit spinor, we get

$$R_1(\psi)\beta_0 = \exp\left[i\sigma_z\psi/2\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \exp\left[i\psi/2\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{6}$$

We now apply the rotation, $R_2 = \exp\left[i\sigma_x\theta/2\right] = \cos\theta 2 + i\sigma_x \sin\theta/2$. We obtain

$$R_2R_1\beta_0 = \exp\left[i\psi/2\right]\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \exp\left[i\psi/2\right]\begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix} . \tag{7}$$

Applying $R_3 = \exp\left[i\sigma_z\varphi/2)\right]$, we get

$$\beta = R\beta_0 = \begin{pmatrix} \cos\theta/2 & \exp\left[i(\psi+\varphi)/2\right] \\ i\sin\theta/2 & \exp\left[i(\psi-\varphi)/2\right] \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} . \tag{8}$$

Writing

$$\beta = \begin{pmatrix} b_1 + ib_2 \\ b_3 + ib_4 \end{pmatrix} , \tag{8-a}$$

[11] For a clear diagram of the Euler angles, see H. GOLDSTEIN: *Classical Mechanics* (Cambridge Mass., 1950), p. 107.

[12] It will become clear that the spinor that we associate to the standard orientation is arbitrary, and that all that is important is the *relationship* among spinors corresponding to different orientations. We have chosen the above standard spinor for the sake of simplicity.

we get

$$(8\text{-}b)\qquad \begin{cases} b_1 = \cos\theta/2\cos(\psi+\varphi)/2\,, & b_2 = \cos\theta/2\sin(\psi+\varphi)/2\,,\\ b_3 = -\sin\theta/2\sin(\psi+\varphi)/2\,, & b_4 = \sin\theta/2\cos(\psi-\varphi)/2\,\cdot \end{cases}$$

The b_i evidently satisfy the relation,

$$(9)\qquad b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1\,.$$

The b_i are just the well-known Cayley-Klein parameters of the rotation group ([13]); of which evidently only three are independent. By specifying the b_i we can solve for the Euler angles, θ, φ, ψ. Indeed, the specification is two-valued, in the sense that there are *two* values of the b_i for each set of Euler angles.

Thus, we see that a unit spinor can be interpreted as defining the orientation of a body in space. We shall therefore tentatively interpret the spinor appearing in the Pauli equation in this way, and then show that a consistent interpretation can so be obtained ([14]).

4. – Canonical Relations for the Cayley-Klein Parameters.

Now, we shall set up a *classical* canonical formalism, in which the Pauli equation is used to define the equations of motion of the Cayley-Klein parameters, and therefore the equations governing the rotations of the bodies making up the fluid.

Our first step is to write down a classical hamiltonian from which the Pauli equation can be derived. This hamiltonian is formally the same function of the spinor Ψ as that which appears in the hamiltonian from which the Pauli equation is derived in the usual form of the quantum theory. This is

$$(10)\qquad H = \int\left[\frac{\hbar^2}{2m}\left|\left(\nabla - ie\frac{A}{c}\right)\Psi\right|^2 + V|\Psi|^2 + \frac{e\hbar}{2mc}\,\Psi^*\boldsymbol{\sigma}\cdot\mathfrak{H}\Psi\right]\mathrm{d}\boldsymbol{x}\,.$$

The above hamiltonian will lead to the correct equation for Ψ provided that we assume the *classical* Poisson bracket relations

$$(11)\qquad [\Psi_a^*(\boldsymbol{x}'),\,\Psi_b(\boldsymbol{x})] = \frac{\delta_{ab}}{i\hbar}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,.$$

([13]) See H. GOLDSTEIN: Chap. 4. Note that we have rotated from body to space axes, while GOLDSTEIN rotates from space to body axes.

([14]) Spinors have already been used to describe the orientation of a rigid body. See, for example, H. B. G. CASIMIR: *Rotation of a Rigid Body in Quantum Mechanics* (The Hague, 1931).

For then we obtain just the Pauli equation

$$i\hbar \frac{\partial \Psi_a}{\partial t} = i\hbar[H, \Psi_a] = -\frac{\hbar^2}{2m}\left(\nabla - ie\frac{\boldsymbol{A}}{c}\right)\Psi_a + V\Psi_a + \frac{e\hbar}{2mc}(\boldsymbol{\sigma}\cdot\mathfrak{H}\Psi)_a\,,$$

where we have used the fact that the Poisson bracket of Ψ with an integral such as H is defined with the aid of the well-known functional derivatives. Thus

$$(12)\qquad [H, F] = \frac{\delta H}{\delta \Psi_1}\frac{\delta F}{\delta \Psi_1^*} + \frac{\delta H}{\delta \Psi_2}\frac{\delta F}{\delta \Psi_2^*} + \frac{\delta H}{\delta \Psi_1^*}\frac{\delta F}{\delta \Psi_1} + \frac{\delta H}{\delta \Psi_2^*}\frac{\delta F}{\delta \Psi_2}\,.$$

In order to simplify the treatment, we now split the whole space into volume elements so small that Ψ does not change appreciably within them. We may then define $\Psi(\boldsymbol{x}_m)$ as the mean value of Ψ in such a region, centering on the point $\boldsymbol{x}_m$. By integrating eq. (11) over small regions of $\boldsymbol{x}$ and $\boldsymbol{x}'$. centered respectively at $\boldsymbol{x}_m$ and $\boldsymbol{x}_n$ and dividing by $(\Delta V)^2$, we get

$$[\Psi_a^*(\boldsymbol{x}_m), \Psi_n(\boldsymbol{x}_n)] = \frac{\delta_{ab}}{i\hbar}\;\frac{\delta(\boldsymbol{x}_m, \boldsymbol{x}_n)}{(\Delta V)}\,.$$

We shall now find it convenient to introduce a new spinor,

$$(13)\qquad \chi = \sqrt{\hbar \Delta V}\,\Psi\,.$$

We obtain

$$(14)\qquad [\chi_a^*(\boldsymbol{x}_m), \chi_b(\boldsymbol{x}_n)] = -i\delta(\boldsymbol{x}_m, \boldsymbol{x}_n)_{a,b}\,.$$

We then write

$$(15)\qquad \chi = \frac{1}{\sqrt{2}}\begin{pmatrix}(a_1 + ia_2)\\ (a_3 + ia_4)\end{pmatrix}.$$

From the Poisson bracket relations, (14), we then deduce that

$$(16)\qquad \begin{cases} [a_1, a_2] = [a_3, a_4] = 1\,,\\ [a_1, a_3] = [a_2, a_4] = [a_2, a_3] = [a_2, a_4] = 0\,. \end{cases}$$

Thus a_1 is the momentum canonically conjugate to a_2 and a_3 is canonically conjugate to a_4. The a_i are evidently proportional to the Cayley-Klein parameters, b_i (given by eq. (8-a)). Thus, we have found the canonical relations among the a_i, implied by the Pauli equation.

5. – Definition of Spin Angular Momentum.

In order to motivate further steps in the interpretation of the Pauli equation, let us recall that in the usual quantum mechanics, the total spin is given by $(\hbar/2)\int\Psi^*\boldsymbol{\sigma}\Psi\,\mathrm{d}\boldsymbol{x}$. This suggests that $(\hbar/2)(\Psi^*\boldsymbol{\sigma}\Psi)$ is a spin density, and that $\boldsymbol{S}=(\hbar/2)(\Psi^*\boldsymbol{\sigma}\Psi)\Delta V$ is the total spin in the small element of volume ΔV. Expressing Ψ in terms of χ through eq. (13), we get

$$\boldsymbol{S}=\tfrac{1}{2}(\chi^*\boldsymbol{\sigma}\chi)\,. \tag{17}$$

The total number of particles in this region is just

$$\Delta N=\Psi^*\Psi\,\Delta V=\chi^*\chi/\hbar\,. \tag{18}$$

Thus, the spin per particle is

$$\boldsymbol{s}=\frac{\hbar}{2}\,\frac{\chi^*\boldsymbol{\sigma}\chi}{\chi^*\chi}=\frac{\hbar}{2}\,\frac{\Psi^*\boldsymbol{\sigma}\Psi}{\Psi^*\Psi}\,. \tag{19}$$

We see then that the maximum spin per body in a given direction is $\hbar/2$. This is in agreement with what one gets for the spin « observable » in the usual quantum theory [15].

With the aid of the P.B. relations (11), it can be shown be means of a simple calculation that the components of the spin vector, $\boldsymbol{S}$, satisfy the cyclical relations, characteristic of angular momenta

$$[S_x,\,S_y]=S_z\,. \tag{20}$$

Thus, the Poisson-Bracket relations obtained in the derivation of the Pauli equations from a hamiltonian are just what are needed to lead to the correct P.B. relations for the angular momenta defined in eq. (17).

[15] In our theory, the spin vector is a *continuous* variable, with arbitrary projections on any axis, and with a total magnitude of $s^2=\hbar^2/4$ (as can easily be proved by using the Pauli identities). On the other hand, the spin « observables » appearing in the usual quantum theory may have only certain discrete projections on any given axis, while the magnitude of the « observable » for the total spin is $\frac{3}{4}\hbar^2$. The origin of this difference arises in the circumstance that in the model that we are proposing here, the spin « observable » is, as we shall see later, a kind of statistical or over-all property of the motions of the bodies in the fluid. Thus, there is no reason why it should be identical with the angular momentum of one of the bodies in the fluid, although there will in general of course exist a certain connection between the « observable » and the spins of the bodies, which we shall discuss in more detail in the subsequent paper.

In the present theory, the basic spinor, $\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix}$ contains only *two* canonically independent variables. We shall now see the physical meaning of this reduction of the number of basic variables to two. To do this, we find it convenient to go to a new set of canonically independent variables which we take to be two of the Euler angles, ψ and φ. In terms of the a_i, these are (as can easily be seen from equ. (8-*b*))

$$\psi = \mathrm{tg}^{-1}\frac{a_2}{a_1} + \mathrm{tg}^{-1}\frac{a_4}{a_3}\,; \qquad \varphi = \mathrm{tg}^{-1}\frac{a_2}{a_1} - \mathrm{tg}^{-1}\frac{a_4}{a_3}\,. \tag{21}$$

By means of a simple calculation, we prove that the quantity canonically conjugate to $\psi/2$ is ϱh, which is proportional to the total number of particles in the region (see equ. (18)), while the quantity canonically conjugate to $-\varphi/2$ is S_z, which is equal to the Z component of the total angular momentum of these particles. Thus, we have

$$\hbar\left[\varrho, -\frac{\psi}{2}\right] = \left[\frac{(a_1^2 + a_2^2 + a_3^2 + a_4^2)}{2}, \left(\mathrm{tg}^{-1}\frac{a_2}{a_1} + \mathrm{tg}^{-1}\frac{a_4}{a_3}\right)\right] = 1\,, \tag{22-a}$$

$$\left[S_z, -\frac{\varphi}{2}\right] = \left[\frac{(a_1^2 + a_2^2 - a_3^2 - a_4^2)}{2}, \left(\mathrm{tg}^{-1}\frac{a_2}{a_1} - \mathrm{tg}^{-1}\frac{a_4}{a_3}\right)\right] = 1\,. \tag{22-b}$$

But now we can show by a simple calculation that φ is also equal to the angle φ' made by the projection of the angular momentum vector $\boldsymbol{S}$ on the X-Y plane. Thus

$$\mathrm{tg}\,\varphi' = \frac{\Psi^*\sigma_y\Psi}{\Psi^*\sigma_x\Psi} = \frac{i\begin{pmatrix} a_1 - ia_2 \\ a_3 - ia_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix}}{\begin{pmatrix} a_1 - ia_2 \\ a_3 - ia_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix}} = \frac{a_2a_4 - a_1a_3}{a_1a_3 + a_2a_4}.$$

Now, from eq. (21)

$$\mathrm{tg}\,\varphi = \left(\frac{a_2}{a_1} - \frac{a_3}{a_4}\right)\Big/\left(1 + \frac{a_2a_3}{a_1a_4}\right) = \frac{a_2a_4 - a_1a_3}{a_1a_3 + a_2a_4}\,.$$

Thus, we see that $\varphi' = \varphi$. Moreover, the co-latitude angle, θ', of the angular momentum vector, is defined by $\cos\theta' = \frac{S_z}{S} = \frac{a_1^2 + a_2^2 - a_3^2 - a_4^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2}$. But from equ. (8-*b*) we can calculate the angle θ, made by the principal axis (1) of the body with the Z axis. This is given by $\cos\theta = \frac{a_1^2 + a_2^2 - a_3^2 - a_4^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2}$. Thus, $\theta' = \theta$.

We conclude then that in the Pauli theory, the angular momentum, $\boldsymbol{S}$, is always pointing along the principal axis (1) of the body. Such a special orientation of the angular momentum cannot be maintained for the most general kind of torque that may act on the body. In the next section and in a subsequent paper, we shall see, however, that the Pauli equation implies a special kind of « quantum-mechanical torque » that permits this condition to be maintained as a consistent subsidiary condition.

The special relation between the direction of $\boldsymbol{S}$ and the principal axis (1) is what permits us to reduce the number of independent canonical variables in the theory. When this relation exists, our theory becomes equivalent to a theory of the angular momentum of a point dipole, which has already been treated by several other authors [16-18]. In our subsequent paper, we shall see, however, that the theory developed here can be generalized to treat cases in which $\boldsymbol{S}$ does not necessarily point along the principal axis.

6. – Relationship Between Spin and Velocity.

Let us now recall that equ. (4-c) defines the velocity of a particle in terms of our spinor. Since the spinor is already interpreted in terms of the orientation of a spinning body, eq. (4-c) clearly implies a certain relationship between the spin and the velocity. We shall now study this relationship and show that it can be understood in terms of a reasonable physical model.

We begin by expressing our spinor (with the aid of eq. (7)) as

$$\Psi = R\begin{pmatrix}\cos\theta/2 \;\; \exp\left[i(\psi+\varphi)/2\right] \\ i\sin\theta/2 \;\; \exp\left[i(\psi-\varphi)/2\right]\end{pmatrix}, \qquad \text{where } R \text{ is real.}$$

Using this value of Ψ in eq. (4-c), we readilly get

$$\boldsymbol{v} = \frac{\hbar}{2m}(\nabla\psi + \cos\theta\,\nabla\varphi) - \frac{e}{c}\boldsymbol{A}\,. \tag{23}$$

The above, is however, an expression of the velocity in a form that has already long been familiar in classical hydrodynamics. Indeed, in classical hydrodynamics [19], it is shown that an arbitrary velocity field can always

([16]) G. F. Uhlenbeck and S. Goudsmith: *Nature*, **117**, 264 (1926).

([17]) H. A. Kramers: *Quantentheorie des Elektrons*, (Leipzig, 1938).

([18]) C. M. Lattes, M. Schenberg and W. Schutzer: *Anais da Academia Brasileira de Ciências*, XIX, no. 3 (September 1947).

([19]) See, for example, H. Lamb: *Hydrodynamics* (Cambridge, 1953), p. 248.

be expressed as

$$\boldsymbol{v} = \nabla S + \xi \nabla \eta - \frac{e}{c} \boldsymbol{A} , \tag{24}$$

where ξ and η are scalars, called the Clebsch parameters, and S is also a scalar.

The curl of the velocity is then

$$\nabla \times \boldsymbol{v} = \nabla \xi \times \nabla \eta - \frac{e}{c} \nabla \times \boldsymbol{A} . \tag{25}$$

(This is quite a different expression from the more common form $\boldsymbol{v} = \nabla S' + \nabla \times \boldsymbol{B} - (e/c)\boldsymbol{A}$, in the sense that S' and S are different functions, since div $(\xi \nabla \eta)$ is not in general equal to zero).

The Clebsch parameters have recently been used by DIRAC [20] and and others [21] to treat a theory of a classical electrified fluid ether. They have also been used by TAKABAYASI [4] to treat possible vortices in the Madelung fluid, and by SCHÖNBERG [7] to treat vortex motions in a more general context.

In order to establish a basis of comparison with the Pauli theory, we shall first treat the ordinary classical hydrodynamics of a charged fluid, capable of maintaining a pressure gradient, in terms of the Clebsch parameters. One of the advantages of these parameters is that they permit the formulation of hydrodynamics in terms of a variational principle. Indeed, if ϱ is the density of the fluid, then the hamiltonian is just equal to the total kinetic plus potential energy of the fluid (including the potential energy due to compression)

$$H_c = \int \left[\frac{\varrho}{2m} \left(\nabla S + \xi \nabla \eta - \frac{e}{c} \boldsymbol{A} \right)^2 + \varrho e \varphi + f(\varrho) \right] \mathrm{d}\boldsymbol{x} , \tag{26}$$

where $\boldsymbol{A}$ is the vector potential, φ, the scalar potential, e/m the ratio of charge to mass for an element of the fluid in question, and $f(\varrho)$ defines the pressure through $P = \partial f / \partial \varrho$.

Now if we adopt essentally the same Poisson Brackets as were derived from the Pauli theory; i.e.

$$[\varrho, S] = \delta(\boldsymbol{x} - \boldsymbol{x}') , \qquad [(\varrho\xi), \eta] = \delta(\boldsymbol{x} - \boldsymbol{x}') , \tag{27}$$

(20) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **209**, 291 (1951); **212**, 330 (1952); **223**, 439 (1954).
(21) See, for example, O. BUNEMAN: *Proc. Camb. Phil. Soc.*, **50**, 77 (1954).

then we obtain the following equations of motion

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \varrho\left(\nabla S + \xi \nabla \eta - \frac{e}{c} \boldsymbol{A}\right) = \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\,, \tag{28-a}$$

$$\frac{\partial S}{\partial t} - \xi(v\cdot\nabla)\eta + \frac{(\nabla S + \xi\nabla\eta - (e/c)\boldsymbol{A})^2}{2m} + \frac{e}{m}\varphi + \frac{\partial f}{\partial \varrho} = 0\,, \tag{28-b}$$

$$\frac{\partial \xi}{\partial t} + (v\cdot\nabla)\xi = \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0\,, \tag{28-c}$$

$$\frac{\partial \eta}{\partial t} + (v\cdot\nabla)\eta = \frac{\mathrm{d}\eta}{\mathrm{d}t} = 0\,. \tag{28-d}$$

Combining (28-*b*) and (28-*d*) we get,

$$\frac{\partial S}{\partial t} + \xi\frac{\partial \eta}{\partial t} + \frac{(\nabla S + \xi\nabla\eta - (e/c)\boldsymbol{A})^2}{2m} + \frac{e}{m}\varphi + \frac{\partial f}{\partial \varrho} = 0\,. \tag{28-e}$$

Eq. (28-*a*) is just the equation of conservation of fluid. Eq. (28-*b*) (and therefore also (28-*e*)) is a generalization of the equation for the velocity potential, S, which enables us to take vorticity into account (through ξ and η), Eqs. (28-*c*) and (28-*d*) are very interesting, for they say that *if we follow a moving fluid element*, ξ and η are constants of the motion (but not at a point fixed in space). Thus, the surfaces, $\xi(x, y, z, t) = \text{const.}$ and $\eta = \eta(x, y, z, t) = \text{const.}$, define the tubes of flow of the fluid.

To understand the motion in more detail, let us focus our attention on eq. (28-*e*). This is analogous to a Hamilton-Jacobi equation, but the equation involves, in addition to the electromagnetic potentials, and the pressure potential ($P = \partial f/\partial\varrho$), a set of equivalent potentials that are functions of the Clebsch parameters [22]. In fact, the Clebsch parameters contribute an addition of $^2(\partial\eta/\partial t)$ to the scalar potential and $-\xi\nabla\eta$ to the vector potential. We may thus conclude that formally they contribute to the equation of motion of a fluid element a « pseudo-Lorentz force », given by

$$F' = \varepsilon' + \boldsymbol{v}\times\mathfrak{H}' \tag{29}$$

where the « pseudo-electric » and « pseudo-magnetic » fields are given

$$\varepsilon' = -\nabla\left(\xi\frac{\partial\eta}{\partial t}\right) + \frac{\partial}{\partial t}(\xi\nabla\eta) = \frac{\partial\xi}{\partial t}\nabla\eta - \frac{\partial\eta}{\partial t}\nabla\xi\,,$$

$$\mathfrak{H}' = -\nabla\times(\xi\nabla\eta) = -\nabla\xi\times\nabla\eta\,.$$

([22]) We are using here an argument due originally to M. SCHÖNBERG: *Nuovo Cimento*, **11**, 674 (1954).

Thus, for the additional force, we get

$$F' = \frac{\partial \xi}{\partial t} \nabla \eta - \frac{\partial \eta}{\partial t} \nabla \xi - \boldsymbol{v} \times (\nabla \xi \times \eta) \,. \tag{30}$$

Writing $-\boldsymbol{v} \times (\nabla \xi \times \nabla \eta) = (\boldsymbol{v} \cdot \nabla \xi) \nabla \eta - (\boldsymbol{v} \cdot \nabla \eta) \nabla \xi$, we obtain

$$F' = \frac{\partial \xi}{\partial t} + (\boldsymbol{v} \cdot \nabla) \xi - \frac{\partial \eta}{\partial t} - (\boldsymbol{v} \cdot \nabla) \eta = \frac{\mathrm{d}\xi}{\mathrm{d}t} \nabla \eta - \frac{\mathrm{d}\eta}{\mathrm{d}t} \nabla \xi \,.$$

Since from (28-c) and (28-d) we have $\mathrm{d}\xi/\mathrm{d}t = 0$ and $\mathrm{d}\eta/\mathrm{d}t = 0$, we get $F' = 0$. Thus the Clebsch parameters formally add to the vector potential a set of quantities that produce a pseudo-electromagnetic field for which the pseudo-Lorentz-force vanishes. Therefore the Clebsch parameters appearing in eq. (28-e) do not alter the equations of motion of a fluid element ([23]). What they do is to enable us to consider in a canonical formalism an ensemble of trajectories for the fluid elements which have a vorticity that does not come from the effects of the electromagnetic vector potential, $\boldsymbol{A}$. Thus, they make possible a generalization of the Hamilton-Jacobi theory; for in the latter, we consider only ensembles in which $m\boldsymbol{v} - (e/c)\boldsymbol{A}$ is derivable from a potential (equal to the action function).

Let us now return to the Pauli theory. Our first problem will be to compare the classical hamiltonian (26) with the Pauli hamiltonian (10). By means of a simple calculation, we obtain the result that the Pauli hamiltonian can be written as the sum of the following three terms:

$$H = H_T + H_{q.} + H_{sp.}$$

where we have

$$H_T = \int \frac{\hbar^2}{2m} \varrho' \left(\frac{\nabla \psi}{2} + \cos\theta \frac{\nabla \varphi}{2} - \frac{e}{c} \boldsymbol{A} \right)^2 \mathrm{d}\boldsymbol{x} \,, \tag{31-a}$$

$$H_{q.} = \int \frac{\hbar^2}{8m} \frac{(\nabla \varrho')^2}{\varrho'} \mathrm{d}\boldsymbol{x} \,, \tag{31-b}$$

$$H_{sp.} = \int \mathrm{d}\boldsymbol{x} \left[\frac{\hbar^2}{8m} \varrho' ((\nabla \theta)^2 + \sin^2\theta (\nabla \varphi)^2) + \right. \\ \left. + \frac{e\hbar}{2mc} (\cos\theta \mathfrak{H}_z + \cos\theta \cos\varphi \mathfrak{H}_x + \cos\theta \sin\varphi \mathfrak{H}_y \right] , \tag{31-c}$$

where $\mathfrak{H}$ is the magnetic field.

([23]) Note that the fact that $\mathrm{d}S/\mathrm{d}t = \mathrm{d}\eta/\mathrm{d}t = 0$ plays an essential role in making this result possible. Later, in connection with spin theory, we are going to have non-zero values for these derivatives; and then, the Clebsch parameters will imply additional terms in the force on a particle.

In the above equations, we have, as in eq. (4), defined $\varrho' = |\Psi|^2 = \varrho$, where ϱ' is the density of bodies.

In order to compare the Pauli hamiltonian to the simple hamiltonian, Hc, of classical hydrodynamics, we note that ϱ' is canonically conjugate to $-\psi/2$, $(\varrho' \cos\theta)$ to $-\varphi/2$. Thus, if we replace ϱ by ϱ', and write

$$\xi = \cos\theta\,, \qquad \eta/\hbar = -\varphi/2\,, \qquad S/\hbar = -\psi/2 \tag{32}$$

then H_T becomes equal to kinetic energy term in eq. (26).

The terms that are left over in the Pauli equation are then $H_{q.}$ and $H_{sp.}$. We shall see in eq. (34-*b*) that $H_{q.}$ leads to the « quantum-potential » term of the Madelung fluid. Thus, the two terms, $H_T + H_{q.}$ represent the energy of a charged Madelung fluid, in which vortex flow as well as potential flow are taking place.

As for the third term, $H_{sp.}$, it is clear first of all that it contains a part describing the energy of a spinning dipole in a magnetic field $\mathfrak{H}$. The magnetic moment of this dipole is $e\hbar/2mc$.

If we assume that our bodies have a radius of the order of the classical electronic radius, $r_0 = 2.8 \cdot 10^{-13}$ cm, then the magnetic moment is $\mu = \lambda e r_0 v/c$ where v is the velocity of rotation of the body at its periphery, and where λ is a constant of the order of unity depending on the charge distribution in the body. We then obtain

$$\frac{v}{c} = \frac{\hbar}{\lambda m c r_0} = \frac{137}{\lambda}\,.$$

Thus, the body must, at its periphery, be moving much faster than light In a non-relativistic theory, such as we are discussing in the present paper, this, of course leads to no difficulties. In a later paper, concerned with the extension of the theory to the Dirac equation, we shall see that the spin motion becomes closely coupled with the mass motion of the fluid, so that a spin angular momentum of $e\hbar/2mc$ for the body is possible, even when the speed of its periphery does not exceed that of light, because the fluid motions are further connected with the body rotations by a spin-orbit coupling, not present in the non-relativistic theory.

As for the remaining part of $H_{sp.}$, this represents an interaction between the directions of the spins of neighboring bodies, which functions essentially as a spin-dependent addition to the « quantum potential ». To obtain a more definite model for this part of the spin energy, we may let δA represent the total angle between two spin vectors, separated by a distance, $\delta \boldsymbol{x}$. Then, by going to spherical polar coordinates in the space of the spin vectors, we can

show that

$$|\delta \boldsymbol{A}|^2 = |\nabla\theta\cdot\delta\boldsymbol{x}|^2 + \sin^2\theta\,|\nabla\varphi\cdot\delta\boldsymbol{x}|^2 .$$

Thus, the energy, $H_{sp.}$ is just an average of $|\delta A|^2$, taken over a small region that surrounds the body and weighted with the density, ϱ. We can therefore interpret $H_{sp.}$ physically as the result of an assumed short-range interaction between bodies, which tends to line up the directions of their spins. For example, the energy of two neighboring bodies could be proportional to

$$U_{int.} = -\cos(\delta A) \cong -1 + (\delta A)^2/2 . \tag{33}$$

Let us now obtain the equations of motion that follow from our hamiltonian. To do this, it will be convenient to write H as

$$H = \int \frac{\varrho'}{2m}\left(\nabla S + \xi\nabla\eta - \frac{e}{c}\boldsymbol{A}\right)^2 \mathrm{d}\boldsymbol{x} + \frac{\hbar^2}{8m}\int\frac{(\nabla\varrho')^2}{\varrho'}\,\mathrm{d}\boldsymbol{x} + \int \varrho' H s\,\mathrm{d}\boldsymbol{\chi},$$

where,

$$H_s = \frac{\hbar^2}{2m}\left[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2\right] + \frac{e}{mc}\,\mathbf{s}\cdot\boldsymbol{\mathfrak{H}} = \frac{\hbar^2}{2m}\left[\frac{(\nabla\xi)^2}{1-\xi^2} + (1-\xi^2)(\nabla\eta)^2\right] + \frac{e}{mc}\,\mathbf{s}\cdot\boldsymbol{\mathfrak{H}} .$$

The equations of motion then become

$$\frac{\partial\varrho'}{\partial t} + \operatorname{div}\varrho'\boldsymbol{v} = 0 , \tag{34-a}$$

$$\frac{\partial S}{\partial t} - \xi(\boldsymbol{v}\cdot\nabla)\eta + \frac{mv^2}{2} - \frac{\hbar^2}{2m}\left(\frac{\nabla^2\varrho'}{\varrho'} - \frac{1}{2}\frac{(\nabla\varrho')^2}{(\varrho')^2}\right) + H_s = 0 , \tag{34-b}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\frac{\delta H_s}{\delta\eta} , \tag{34-c}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\delta H_s}{\delta\xi} . \tag{34-d}$$

Eq. (34-a) is then just the conservation equation while eq. (34-b) is the pseudo Hamilton-Jacobi equation for a fluid with the usual quantum potential, $-\left(\frac{\hbar^2}{2m}\frac{\nabla^2\varrho'}{\varrho'} - \frac{\hbar^2}{4m}\frac{(\nabla\varrho')^2}{(\varrho')^2}\right)$, and a spin-dependent quantum potential, $H_s - \xi(\boldsymbol{v}\cdot\nabla)\eta$.

Eqs. (34-c) and (34-d), which tell us how the spin directions change with time are in precisely the same form as that of the equations expressing the

torque acting on a point dipole, derived previously by SCHÖNBERG ([18]) in a purely classical problem.

We may now, however, obtain another instructive form of the equations for the spin variables by noting that $H_{sp.}$ can be written in the form

$$H_{sp.} = \frac{\hbar^2}{2m}\sum_{i,j}\left(\frac{\partial s_i}{\partial \chi_j}\right)^2 + \frac{e}{mc}\,\boldsymbol{s}\cdot\boldsymbol{\mathfrak{H}}\,. \tag{35-a}$$

Noting that $\varrho = \dfrac{2|S|}{\hbar}$, we also obtain the expression

$$H_{q.} + H_{sp.} = \int\left[\frac{\hbar^2}{2m}\sum_{i,j}\left(\frac{\partial S_i}{\partial \chi_j}\right)^2\Big/\varrho + \frac{e}{mc}\,(\boldsymbol{S}\cdot\boldsymbol{\mathfrak{H}})\right]\mathrm{d}\lambda\,. \tag{35-b}$$

We can obtain the equations of motion for $\boldsymbol{s}$, by writing $\pi = \boldsymbol{S}/\varrho$, and using the P.B. relations for the angular momentum. We get (with the aid of eq. (35-a))

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = -\frac{\boldsymbol{s}}{\varrho}\times\sum_i\frac{\partial}{\partial x_i}\left(\varrho\,\frac{\partial \boldsymbol{s}}{\partial x_i}\right) + \frac{e}{mc}\,(\boldsymbol{s}\times\boldsymbol{\mathfrak{H}})\,. \tag{36}$$

Thus, the spin vector precesses about the magnetic field, with angular frequency, $e\mathfrak{H}/mc$, and there is an additional precession with angular velocity $-\dfrac{1}{\varrho}\sum_i\dfrac{\partial}{\partial x_i}\left(\varrho\,\dfrac{\partial s}{\partial x_i}\right)$, which results from the torques produced by the neighboring spins. In some ways, one may think of this extra torque as due to a kind of « quantum-mechanical » addition to the magnetic field. But to make the analogy correct, one must think of the magnetic field in a polarizable medium where each dipole interacts strongly with its neighbors.

Finally, we shall discuss the motion of an element of fluid. This is best done in terms of the expression for the energy-momentum-stress tensor. To derive this tensor, we start with the lagrangian leaving out the vector and scalar potentials, which do not alter the problem in any essential way

$$L = \int \varrho'\left[\frac{\partial S}{\partial t} + \xi\,\frac{\partial \eta}{\partial t} - \frac{\hbar^2}{8m}(\nabla\psi + \xi\,\nabla\eta)^2 - \frac{\hbar^2}{8m}\,\frac{(\nabla\varrho')^2}{(\varrho')^2} - \right.$$
$$\left. - \frac{\hbar^2}{8m}\left((\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2\right)\right]\mathrm{d}\boldsymbol{x}\,. \tag{37}$$

It can easily be shown that this Lagrangian leads to the correct canonical relations (27) between ξ and η, and between ϱ' and S, and that it leads to the Pauli hamiltonian (10).

Now the canonical energy-momentum stress tensor is given by [24]

$$T_{ij} = \sum_\alpha \frac{\partial L}{\partial(\partial f_\alpha/\partial x_i)} \frac{\partial f_\alpha}{\partial x_j}\,; \qquad T_{0j} = \sum_\alpha \frac{\partial L}{\partial(\partial f_\alpha/\partial t)} \frac{\partial f_\alpha}{\partial x_j}\,;$$

$$T_{00} = \sum_\alpha \frac{\partial L}{\partial(\partial f_\alpha/\partial t)} \frac{\partial t_\alpha}{\partial f}\,,$$

where the f_α represent all the possible field quantities. T_{0j} represents the momentum density, and T_{00} the energy density. We readily obtain

$$\text{(38)} \quad \left\{ \begin{aligned} T_{00} &= \varrho'\left(\frac{\partial S}{\partial t} + \xi\frac{\partial\eta}{\partial t}\right), \\ T_{0j} &= \varrho'\left(\frac{\partial S}{\partial x_0} + \xi\frac{\partial\eta}{\partial x_j}\right), \\ T_{ij} &= \frac{\varrho'}{m}\left(\frac{\partial S}{\partial x_i} + \xi\frac{\partial\eta}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_j} + \xi\frac{\partial\eta}{\partial x_j}\right) + \frac{\hbar^2}{4m}\frac{\partial\varrho'}{\partial x_i}\frac{\partial\varrho'}{\partial x_j} + \\ &\qquad + \frac{\hbar^2}{4m}\varrho'\left(\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} + \sin^2\theta\,\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\right), \end{aligned} \right.$$

where ξ and η are given by eq. (32).

The energy per particle is then $\dfrac{\partial S}{\partial t} + \xi\dfrac{\partial\eta}{\partial t}$, which agrees with what we obtain with the « Hamilton-Jacobi » equation (28-e). The momentum per particle is just $\boldsymbol{p} = \nabla S + \xi\nabla\eta$, in agreement with what we have been assuming. The stress tensor, T_{ij}, contains three parts. The first part is just $m\varrho' v_i v_j$, which is just the usual term representing the effects of mass motion. The second part corresponds to the quantum potential and the third part to the effects of the interactions between the spins of neighboring particles. The equation of conservation of momentum can then be written as

$$\frac{\partial}{\partial t}(\varrho' v_i) + \sum_j \frac{\partial T_{ij}}{\partial x_j} = 0\,.$$

With the aid of the equation for conservation of particles, and the relation

$$\sum_j \frac{1}{\varrho'}\frac{\partial}{\partial x_j}\left(\frac{1}{\varrho}\frac{\partial\varrho'}{\partial x_i}\frac{\partial\varrho'}{\partial x_j}\right) = \frac{\partial}{\partial x}\left(\frac{\nabla^2\varrho^{\frac12}}{\varrho^{\frac12}}\right) = \frac{\partial U}{\partial x_i}\,,$$

we obtain

$$\text{(39)} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla U - \frac{1}{\varrho}\frac{\partial}{\partial x_j}(\varrho S_{ij})\,,$$

[24] See, for example, G. WENTZEL: *Quantum Theory of Fields* (New York, 1949).

where

$$S_{ij} = \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} + \sin^2\theta\,\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,.$$

We see then that the fluid element moves under the action of the « quantum-mechanical » stress tensor, $\dfrac{\hbar^2}{4m\varrho'}\dfrac{\partial\varrho'}{\partial x_i}\dfrac{\partial\varrho'}{\partial x_j}$, which leads to the quantum potential, and an additional quantum-mechanical contribution to the stress tensor, arising from the interaction of neighboring spins.

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A Causal Interpretation of the Pauli Equation (B).

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CONTENTS:

1. – Introduction.

In a previous paper (1), to which we shall hereafter refer as (A), we have developed a causal interpretation of the Pauli equation, in terms of the model of a fluid composed of spinning bodies, in which the spin angular momentum of each body is directed along its first principal axis (i.e., its axis of symmetry). The object of the present paper is two-fold, first, to generalize the above model to show how the Pauli equation results as a special case of a theory in which the angular momentum may point in an arbitrary direction, and secondly, to illustrate our model of the Pauli theory in terms of the simple example of a stationary state of an electron in an atom, and to apply this example in the theory of measurements.

In Secs. 2 and 3 we shall develop our more general model and show that in terms of it, the Pauli equation follows as a consistent subsidiary condition. In other words, if the angular momentum points along the first principal axis at any instant, say $t = 0$, then the torques will be such that this condition will be maintained for all time. Moreover, small deviations from this condition lead to a rapid oscillation of the motion about a mean, in which the

(1) D. BOHM, R. SCHILLER and R. TIOMNO: *Nuovo Cimento*, 1, 48 (1955).

condition is satisfied. Physically, such an oscillatory motion corresponds to a precession of the angular momentum about the direction of the axis of symmetry, in which the component of the angular momentum perpendicular to this axis averages out to zero. Thus, even in the case of a general orientation of the angular momentum vector, the Pauli equation would still be a good approximation for processes that are slow compared with the rate of precession described above, while for more rapid processes, it would cease to be a good approximation. Instead, a more general equation (which will be seen to be non-linear) would have to be used. Thus, we are led to a specific example in which the causal interpretation of the quantum theory implies new kinds of equations in connection with high frequencies (and therefore high energies), equations which would reduce approximately to those of the usual quantum theory only at those relatively low energies for which the usual theory is known to be valid.

In Sec. **4**, we then illustrate our model of the Pauli equation in terms of a stationary state of an electron in an atom. This illustration serves not only to bring out important topological properties of a field of body orientations, such as is implied by our model, but it also shows how a continuous distribution of spin angular momentum throughout the fluid can lead to discrete or « quantized » possible values for the angular momentum of the fluid as a whole. Quantization of angular momentum (and of other variables) is therefore seen to arise as an over-all property of the fluid, on the basis of a lower level of continuous motion, much as discrete frequencies of stationary modes of vibration arise in connection with the motions of continuous fields in classical mechanics.

In Sec. **5**, we indicate how the process of measurement is to be treated in terms of our model. What is observed in a measurement of the angular momentum « observable » will be seen to be just one of the discrete possible stationary values for the total angular momentum of the system, and not the continuously distributed angular momenta existing in the various parts of the fluid.

Finally, in Sec. **6**, we summarize the essential features of the model, and suggest possible further directions of research.

2. – Treatment of Arbitrary Direction of Spin Motion for a Fixed Rigid Body.

Our first step will be to extend to the case of arbitrary spin motions the hamiltonian formalism in terms of spinor variables developed in paper (A).

To do this, we begin with the case of a rigid body at rest. In the next section, we shall then extend the theory to the more general case of a field of rigid spinning bodies with translational motions.

The lagrangian of a rigid body with no translational motion is just the kinetic energy due to its spin, which is (2)

$$L = \tfrac{1}{2}\sum_{i,k} I_{ik}\omega_i\omega_k\,, \tag{1}$$

where the ω_i represent the components of the angular velocity, and I_{ik} is the tensor for the moment of inertia of the body. The components of the angular momentum are then given by

$$s_i = \sum_k I_{ik}\omega_k \tag{2}$$

so that

$$L = \tfrac{1}{2}\sum_i s_i\omega_i\,. \tag{3}$$

The next problem is to express the ω_i in terms of spinors. Now, the relationship between the ω_i and the Euler angles is (3)

$$\left\{\begin{aligned} \omega_z &= \dot\psi\cos\theta + \dot\varphi\,,\\ \omega_x &= \dot\psi\sin\theta\sin\psi + \dot\theta\cos\psi\,,\\ \omega_z &= \dot\psi\sin\theta\cos\psi - \theta\sin\psi\,. \end{aligned}\right. \tag{4}$$

We now return to our spinor $\beta = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$, defined in paper (A), eq. (7). We have

$$\dot\beta_1 = \exp\left[\frac{i(\psi+\varphi)}{2}\right]\left(\frac{i}{2}(\dot\psi+\dot\varphi)\cos\frac{\theta}{2} - \frac{\dot\theta}{2}\sin\frac{\theta}{2}\right)$$

$$\dot\beta_2 = i\exp\left[\frac{i(\psi-\varphi)}{2}\right]\left(\frac{i}{2}(\dot\psi-\dot\varphi)\sin\frac{\theta}{2} + \frac{\dot\theta}{2}\cos\frac{\theta}{2}\right).$$

Now let us consider the quantity

$$\beta^*\sigma_z\dot\beta - \dot\beta^*\sigma_z\beta = i\left(\cos^2\frac{\theta}{2}(\dot\psi+\dot\varphi) - \sin^2\frac{\theta}{2}(\dot\psi-\dot\varphi)\right) = i(\dot\psi\cos\theta+\dot\varphi)\,.$$

Hence we obtain

$$\omega_z = i(\dot\beta^*\sigma_z\beta - \beta^*\sigma_z\dot\beta)\,.$$

(2) In the present paper, we leave out the electromagnetic field, which makes no essential change in the formulation of the theory.

(3) H. Goldstein: *Classical Mechanics* (Cambridge Mass., 1950). See Chap. 4.

And more generally, because of rotational symmetry

$$\boldsymbol{\omega} = i(\dot{\beta}^*\boldsymbol{\sigma}\beta - \beta^*\boldsymbol{\sigma}\dot{\beta}) . \tag{5}$$

The kinetic energy then takes the form

$$L = T = \frac{i}{2}\left(\dot{\beta}^*(\boldsymbol{\sigma}\cdot\boldsymbol{s})\beta - \beta^*(\boldsymbol{\sigma}\cdot\boldsymbol{s})\dot{\beta}\right) . \tag{6}$$

We now define the momenta canonically conjugate to the components of the spinors β and β^*, which are

$$p_1 = \frac{\partial L}{\partial\dot{\beta}_1}, \quad p_2 = \frac{\partial L}{\partial\dot{\beta}_2}, \quad p_1^* = \frac{\partial L}{\partial\dot{\beta}_1^*}, \quad p_2 = \frac{\partial L}{\partial\dot{\beta}_2^*} . \tag{7}$$

We also define the spinors

$$\pi = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \pi^* = \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix}, \tag{8-a}$$

so that formally speaking

$$\pi = \frac{\partial L}{\partial\dot{\beta}}, \quad \pi^* = \frac{\partial L}{\partial\dot{\beta}^*} . \tag{8-b}$$

We then obtain from (6)

$$\pi = -i(\sigma\cdot\boldsymbol{s})\beta^* , \tag{9-a}$$

$$\pi^* = i(\boldsymbol{\sigma}\cdot\boldsymbol{s})\beta . \tag{9-b}$$

Note that in eq. (6), $\boldsymbol{s}$ is a function of $\dot{\beta}$ given by eqs. (2) and (5). As can easilly be verified, however, this dependence merely leads to a multiplication by a factor of two of the result that would be obtained if $\boldsymbol{s}$ were not a function of $\dot{\beta}$.

We readilly see that under rotations, π transforms contragrediently to β. To prove this, we form the quantity $\pi\beta$. From eq. (9-a), we obtain

$$\pi\beta = - i(\beta^*(\boldsymbol{\sigma}\cdot\boldsymbol{s})\beta) ,$$

which is evidently a scalar, so that π must transform in the same way as β^*, or in other words, contragrediently to β.

To solve for s_i, we multiply (9-*a*) by $\beta\sigma_i$, and (9-*b*) by $\beta^*\sigma_i$. We obtain

$$\boldsymbol{s} = \frac{i}{2}(\pi\boldsymbol{\sigma}\beta - \pi^*\boldsymbol{\sigma}\beta^*). \tag{10}$$

We shall now find it convenient to obtain the expressions for the projections of the angular momentum vector on the principal axes of the body. It is evident that each of these projections is a scalar with regard to rotations of the space axes, since by definition, these components will not depend on which space axes we choose for the expression of $\boldsymbol{s}$. We can therefore evaluate the components of the angular momentum along the body axes by choosing our space-axes to agree with the body axes. In this case, according to paper (A), eq. (7), we have $\beta = \begin{pmatrix}1\\0\end{pmatrix}$, and for the components of the angular momenta, we get

$$T_1 = i\frac{(\pi_1 - \pi_1^*)}{2}, \quad T_2 = i\frac{(\pi_2 - \pi_2^*)}{2}, \quad T_3 = \frac{(\pi + \pi^*)}{2}. \tag{11-a}$$

We also introduce for convenience, the quantity, T_0, which as we shall see presently is zero in our problem,

$$T_0 = \frac{\pi_1 + \pi_1^*}{2}. \tag{11-b}$$

We wish now to express the T_i in terms of the spinor quantities, taken in an arbitrary frame of reference. To do this, we seek a set of scalar functions of the spinors π and β which reduce to the T_i when we chose the space frame to be the same as the body frame.

Let us first consider the scalar $\pi\beta$. In the frame in which $\beta = \begin{pmatrix}1\\0\end{pmatrix}$, this reduces to

$$\pi\beta = \pi_1 = T_0 - iT_1,$$

so that

$$T_1 = 2\frac{(\pi\beta - \pi^*\beta^*)}{2}, \tag{12-a}$$

$$T_0 = \frac{\pi\beta + \pi^*\beta^*}{2}. \tag{12-b}$$

Now from eq. (9-*b*), we see (since $\boldsymbol{s}$ is real) that $T_0 = 0$. This relationship is a subsidiary condition whose meaning can be seen by writing β and π in

extended form

$$\beta = \begin{pmatrix} b_1 + ib_2 \\ b_3 + ib_4 \end{pmatrix}, \quad \pi = \begin{pmatrix} p_1 - ip_2 \\ p_3 - ip_4 \end{pmatrix},$$

where p_i is canonically conjugate to b_i. We then get

$$\pi\beta + \pi^*\beta^* = b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 \,. \tag{1)3}$$

If we consider the space of the Cayley-Klein parameters introduced in paper (A), eq. (8-b), we see that from the relation $b_1^2+b_2^2+b_3^2+b_4^2=1$, it follows that we must remain on a unit hyper-sphere in this four dimensional space. Now eq. (13) then expresses the fact that the « radial momentum » in this hyper-space is zero. Such a result is to be expected, since the kinetic energy is a function only of the angular velocities, which do not involve the « radial » component of the velocity in the C.K. space, so that the corresponding momentum must be zero.

From (11-a) and (11-b), we readilly obtain for the total spin

$$S^2 = T^2 = (\pi^*\pi)^2 - T_0^2 = (\pi^*\pi)^2 \qquad (\text{since } T_0 = 0). \tag{14}$$

We now wish to obtain expressions corresponding to (12-a) and (12-b) for T_2 and T_3. To do this, we first consider the spinor

$$\tilde{\beta} = \begin{pmatrix} -\beta_2^* \\ \beta_1^* \end{pmatrix}. \tag{15}$$

The spinor $\tilde{\beta}$ has the same transformation properties under rotation as does β itself, as can be readilly verified by direct computation with the transformation matrix $\exp[i(\boldsymbol{R}\cdot\boldsymbol{\sigma})/2]$. Indeed, we have $\tilde{\beta} = i\sigma_y\beta^*$, which is just the non-relativistic form of the so-called charge conjugate spinor [4] which, as is well-known, has the same transformation properties as the spinor β itself.

We now consider the invariant $\pi\tilde{\beta}$. If we evaluate this in a frame in which the space and body axes are the same, so that $\tilde{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we obtain (using eq. (11-a))

$$\pi\tilde{\beta} = \pi_2 = T_3 - iT_2 \,, \tag{16-a}$$

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[4] W. PAULI: *Rev. Mod. Phys.*, 13, 203 (1941).

or

$$T_3 = \frac{\pi\tilde{\beta} + \pi^*\tilde{\beta}^*}{2}, \qquad T_2 = i\,\frac{(\pi\tilde{\beta} - \pi^*\beta^*)}{2}. \tag{16-b}$$

It is readilly verified by direct computation that (T_1, T_2, T_3) satisfy the characteristic P.B. relationships of angular momenta, $(T_1, T_2) = T_3$, etc.. One also readily verifies that the P.B. of any component of T, with any component of S are zero, as they have to be, since S is the generator of the infinitesimal rotation of the space axes and since the T_i are scalars with respect to such rotations.

We shall now go to the hamiltonian formalism. The hamiltonian is

$$H = \sum_i s_i w_i - L = \sum \frac{s_i w_i}{2} = L\,. \tag{17}$$

We must now eliminate the ω_i from the hamiltonian. This is most conveniently done by expressing the angular momentum in terms of its projection T_i on the principal axes of the body. This gives

$$H = \frac{T_1^2}{2I_1} + \frac{T_2^2}{2I_2} + \frac{T_3^2}{2I_3}, \tag{18}$$

where I_1, I_2, I_3, are the moments of inertia relative to the principal axes. For the case of a symmetrical top, which interests us here, $I_2 = I_3 = I$. Let us write $I_1 = I/\varepsilon$. Eq. (18) then becomes

$$H = \frac{1}{2I}(\varepsilon T_1^2 + T_2^2 + T_3^2) = \frac{1}{2I}(T^2 + (\varepsilon - 1)T_1^2)\,. \tag{19}$$

To obtain the equations of motion for S, we note that the P.B.'s of S with the T_i are zero. Thus the components of the angular momentum, taken with respect to axes fixed in space, are constants of the motion. As for the T_i, we note that T_1 is also a constant of the motion. For T_2 and T_3, we readilly obtain the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_3 - iT_2) = i\,\frac{(\varepsilon - 1)}{I}\,T_1(T_3 - iT_2)\,. \tag{20}$$

Thus, the component of the angular momentum normal to the first principal axis rotates with angular velocity, $\omega = ((\varepsilon - 1)/I)T_1$. This means that the first principal axis rotates around the direction of the vector, s, with the angular velocity ω. Thus, on the average the first principal axis points in

the direction of s; and insofar as phenomena that do not change much in the time $\tau = 1/\omega$ are concerned, the body acts as if its angular momentum were parallel to this principal axis.

We note that when $\varepsilon < 1$ (disc-shaped object), the energy is a minimum (for a given total angular momentum $|T|$) when $T_1^2 = T^2$; i.e., when the principal axis points along the direction of s. Thus, for this case, the motion in which the principal axis is parallel to s will be *stable*. If $\varepsilon > 1$, (cigar-shaped object), such motion becomes unstable.

Finally, we shall express in terms of spinors the condition that the motion reduces to the type implied by the Pauli equation, in which the spin points along the principal axis. If this condition is satisfied, we have $T_2 = T_3 = 0$; or $\pi\widetilde{\beta} = 0$. This leads to

$$\pi_1\beta_2^* = \pi_2\beta_1^* ; \qquad \text{or} \qquad \pi_1/\pi_2 = \beta_1^*/\beta_2^* ,$$

so that

$$\pi = ik^2\beta^* , \tag{21}$$

where k is a constant, and where we have chosen the factor of i to simplify subsequent expressions. But in order that $T_0 = (\pi\beta + \pi^*\beta^*)/2 = 0$, k^2 must be a real number. If this is the case, however, then we see that $i\beta^*$ is canonically conjugate to β, which is what is essentially the same relation as paper (A) eq. (14), which holds in the Pauli theory. To demonstrate in more detail the relationship between π, β, and the Pauli spinor, we make the following canonical transformation to the spinors A and B

$$\begin{cases} A = (k\beta + i\pi^*/k)1/\sqrt{2} , \\ B = (k\beta - i\pi^*/k)1/\sqrt{2} . \end{cases} \tag{22}$$

From the above, we readilly prove the Poisson-bracket relations

$$[A, B] = [A^*, B^*] = 0 ; \qquad [A^*, A] = [B^*, B] = i . \tag{23}$$

To satisfy eq. (21), we choose $B = 0$. Then the spinor A^* has the same P.B.'s with A as does the Pauli spinor (see paper (A), eq. (14)). Moreover, we also have $A = k\beta$, so that β is proportional to the Pauli spinor. To express the angular momentum, we eliminate π and β from eq. (10), obtaining

$$\boldsymbol{s} = k(A^*\boldsymbol{\sigma}A) . \tag{24}$$

Thus, the theory is able to treat an arbitrary value of the spin angular momentum. To obtain the same value as in the Pauli theory we must set $k=\hbar/2$. In this way, we complete our demonstration showing how the canonical formalism for an arbitrary direction of spin reduces to that of the Pauli theory when the spin points along the principal axis.

3. – Extension to Rigid Bodies with Translational Motion.

In the previous section, we developed a hamiltonian formalism that permits a treatment in terms of spinors of arbitrary motions of a rigid body fixed in space. We shall now extend this formalism to permit the treatment of a field of such rigid bodies, undergoing translation through space.

To do this, we shall first write the kinetic energy of translation of the bodies in terms of Clebsch parameters as we did in the case of the Pauli theory. Now, in the Pauli theory, we found that one rather naturally obtained the expression given in paper (A), eq. (25) for the velocity, $\boldsymbol{v} = \hbar/2(\nabla\psi+\cos\theta\,\nabla\varphi)$, which involved only the pair of Clebsch parameters, $\cos\theta$ and $\varphi/2$, along with the velocity potential, $\psi/2$. It is possible, however, to introduce as many additional pairs of Clebsch parameters as we please in the definition of the velocity. Thus we may write

$$\boldsymbol{v} = \nabla\lambda + \sum_i \xi_i \nabla\eta_i\,. \tag{25}$$

Now, it is true that the most general velocity field can in principle always be expressed without these additional pairs of Clebsch parameters. Nevertheless, the physical conditions of the problem may often make it convenient to introduce such additional pairs. Indeed, as we shall see presently, the treatment of the problem of the motion of a body with a general orientation of its angular momentum relative to its principal axes is an example of just such a problem.

In terms of the expression (25) for the velocity, the kinetic energy of translation of the bodies then becomes

$$T = \frac{\varrho}{2m}\,(\nabla\lambda + \sum_i \xi_i\,\nabla\eta_i)^2\,\mathrm{d}\boldsymbol{x}\,. \tag{26}$$

Now, introducing the P.B. relations

$$[\varrho(\boldsymbol{x}),\,\lambda(\boldsymbol{x}')] = \big[(\varrho(\boldsymbol{x})\xi_i(\boldsymbol{x})),\,\eta_i(\boldsymbol{x}')\big] = \delta(\boldsymbol{x}-\boldsymbol{x}') \tag{27}$$

with all other P.B.'s zero, we get for the equations of motion

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\,, \tag{28-a}$$

$$\frac{\partial \lambda}{\partial t} - \sum_i \xi_i (\boldsymbol{v}\cdot\nabla)\frac{\eta_i}{m} + \frac{v^2}{2m} = 0\,, \tag{28-b}$$

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \frac{\mathrm{d}\eta_i}{\mathrm{d}t} = 0\,. \tag{28-c}$$

Eq. (28-a) expresses just the conservation of bodies, while (28-b) is an extension of the Hamilton-Jacobi equation. Eq. (28-c) expresses the constancy of the Clebsch parameters as we follow a moving body. It is this property of the Clebsch parameters that is most interesting to us here; for eq. (28-c) means that the ξ_i and the η_i may represent internal properties of bodies (such as angle or spin variables). While these variables remain constant in eq. (28-c), if we add terms to the hamiltonian involving the ξ_i and the η_i, then the equation that takes the place of (28-c) will tell how these properties change *as we follow the moving body.*

The Clebsch parameters therefore provide a natural canonical treatment of the motions of a field of parameters that represent internal properties of moving bodies.

Now, in the Pauli theory, the current vector was expressed as

$$\boldsymbol{j} = \varrho\,\nabla\lambda + (\varrho\xi)\,\nabla\varphi\,.$$

with

$$\xi = \cos\theta/2\,, \qquad \eta = -\,\varphi/2\,, \qquad \lambda = -\,\psi/2\,.$$

Let us recall that $\hbar\varrho$ was canonically conjugate to $-\,\psi/2$, $\hbar\varrho\xi$ to $-\,\varphi/2$. Thus writing $\hbar\varrho/2 = J_\psi$, $\hbar\varrho\cos\theta/2 = J_\varphi$, we have (remembering that J_ψ is density of the component of the angular momentum in the direction of the principal axis, and J_φ the density of the Z component of the angular momentum)

$$\boldsymbol{j} = J_\psi\,\nabla\psi + J_\varphi\,\nabla\varphi\,.$$

But now we are going to give up the restriction that $\boldsymbol{S}$ is directed along the principal axis. To complete the description of the angular momentum, we shall introduce, J_θ, the momentum canonically conjugate to the Euler angle, θ. This is equal to the density of angular momentum about an axis perpendicular to the plane containing the Z axis and the principal axis (1) of the body. In order that the P.B.'s of the kinetic energy of translation with J_θ and θ

shall give $\mathrm{d}J_\theta/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ respectively, it is clearly necessary, however, that $\boldsymbol{j}$ shall be the same functions of J_θ and θ as it is of J_φ and φ, and of J_ψ and ψ. This, we write

$$\boldsymbol{j} = J_\psi \nabla\psi + J_\varphi \nabla\varphi + J_\theta \nabla\theta \,.$$

Now, in order to obtain a simple model that approaches the Pauli case, we set the density of bodies equal to $2J_\psi/\hbar$, which is proportional to the density of the component of the angular momentum along the first principal axis. This implies that all bodies have the same value of this component of the angular momentum. We shall see presently that this assumption is consistent with the equations of motion that we are going to adopt. We then get

$$\boldsymbol{v} = \frac{\boldsymbol{j}}{\varrho} = \frac{\hbar}{2}\,(\nabla\psi) + \frac{J_\varphi}{\varrho}\,\nabla\varphi + \frac{J_\theta}{\varrho}\nabla\theta \,. \tag{27}$$

The next step is to add the kinetic energy of spin to the hamiltonian. To do this, we introduce the quantities $Q_i = \varrho T_i$, which are the densities of the components of the angular momenta taken along the principal axes. From the results of the previous section, this is

$$H_1 = \int \frac{1}{2\varrho T}\,(Q^2 + (\varepsilon - 1)Q_1^2)\,\mathrm{d}\boldsymbol{x} \tag{28}$$

(note that by definition, $Q_1 = J_\psi$).

Finally, we must add an appropriate generalization of the quantum mechanical part of the energy (« quantum potential » plus spin potential), which, for the Pauli case is expressed in paper (A), eq. (35-*b*). We write for the term

$$H_2 = \frac{\hbar^2}{2m}\int \sum_{ij}\left(\frac{\partial S_i}{\partial x_j}\right)^2 \mathrm{d}\boldsymbol{x} \,. \tag{29}$$

The complete hamiltonian is then

$$H = H_r + H_1 + H_2 \,. \tag{30}$$

Let us now obtain some of the equations of motion. First of all, since Q_1 has zero P.B.'s with S_i, we get for $Q_1 = J_\psi$,

$$\frac{\partial Q_1}{\partial t} + \operatorname{div}\,(Q_1 \boldsymbol{v}) = 0 \,. \tag{31}$$

This is just a conservation equation, for the component of the angular momentum along the first principal axis. Since the density of bodies, ϱ, satisfies the same equation, we see that if we choose $\hbar\varrho/2 = Q_1$ at any instant, say $t = 0$, they will remain equal for all time. But $\hbar\varrho/2 = Q_1$ implies that all of the bodies have the same value of the component of the angular momentum along the principal axis. Since this component is a constant of the motion, the condition that it has the same value for all the bodies will evidently likewise be maintained for all time, if it is satisfied at any time.

As for Q_2 and Q_3, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{Q_3 - iQ_2}{\varrho}\right] = \frac{(\varepsilon - 1)Q_1}{I\varrho}\,\frac{Q_3 - iQ_2}{\varrho}\,. \tag{32}$$

Thus, as in the case of the freely rotating body, the component of the angular momentum perpendicular to the first principal axis rotates with angular velocity $\omega = ((\varepsilon - 1)Q_1)/I\varrho$. We may evidently then adopt $Q_2 = Q_3 = 0$ as a consistent subsidiary condition, so that if the body is set with $\boldsymbol{S}$ originally in the direction of its first principal axis, then the torques will be such that this condition is maintained for all time. Moreover, if the principal axis is nearly in the direction of the angular momentum, then Q_2 and Q_3 will rotate with angular velocity, ω, so that for phenomena involving characteristic times τ, which are much less than $1/\omega$, the body will act as if $\boldsymbol{S}$ were parallel to the first principal axis.

Since $Q_2 = Q_3 = 0$ is a consistent subsidiary condition, we may obtain the equations of motion of the remainder of the variables by setting $Q_2 = Q_3 = 0$ in the hamiltonian. The parts, $H_T + H_2$, then reduce to the Pauli hamiltonian (see paper (A) eqs. (31) and (35)). The term, H_1 becomes

$$H_1 = \int \frac{\varepsilon Q_1^2}{2\varrho I}\,\mathrm{d}\boldsymbol{x} = \frac{\hbar^2\varepsilon}{8I}\int \varrho\,\mathrm{d}\boldsymbol{x}\,,$$

where we have written $Q_1 = \hbar\varrho/2$. This part of the hamiltonian will have the same effect as adding to the Hamilton-Jacoby equation (paper (A), eq. (34-*b*)) a « rest mass » term such as would be obtained by deducing the Pauli equation as the non-relativistic limit of the Dirac equation. To bring out this property of H_1 more directly, we may make the canonical transformation to the Pauli spinor, A, given in eq. (22), which yields the « rest mass » term,

$$H_1 = \frac{\hbar^2\varepsilon}{8I}\int (A^* A)\,\mathrm{d}\boldsymbol{x}\,. \tag{33}$$

We conclude then that if the subsidiary condition, $Q_2 = Q_3 = 0$ is satisfied, our set of equations representing the motion of a field of spinning bodies

will reduce essentially to the Pauli equation, which is a *linear* equation for a spinor, Ψ. Moreover, even if the spin does not point along the principal axis, the components Q_2 and Q_3 will in general turn with some angular velocity, ω, so that for processes involving characteristic times, $\tau \gg 1/\omega$, the Pauli equation will still serve as an adequate approximation, since the effects of the components Q_2 and Q_3 will average out to zero in such processes. But in processes so rapid that $\tau \simeq 1/\omega$, the Pauli equation would in such cases cease to be a good approximation, and one would have to go back to our basically non-linear set of equations for the various spin and angle variables. Of course, in the present theory, there is nothing to determine the value of ω. We may, however, consistently suppose that $1/\omega$ is of the order of the times involved in the high energy processes connected with the « creation », « destruction » and « transformation » of so-called « elementary » particles. In this case, the linear Pauli equation would be a good approximation for atomic processes, but would break down completely with regard to the high energy processes described above. Of course, to treat such processes correctly, we should need a relativistic theory, and this will require a causal interpretation of the Dirac equation. But we can already see that the generalization of the Pauli equation adopted here leads in a natural way to a breakdown in connection with high energy processes of the hypothesis of linear superposition, which is one of the basic postulates underlying the present form of the quantum theory. When this postulate fails, then the usual interpretation of the quantum theory cannot consistently be applied. This is an example of how a causal interpretation of the quantum theory permits the consideration of new kinds of theories, not permitted if one restricts oneself to the theories that are consistent with the usual interpretation.

Finally, we note that in the present theory, the basic meaning of $\hbar$ is that $\hbar/2$ is the angular momentum per body in the fluid. Thus, in the theory proposed here, the statement that $\hbar$ is a universal constant of nature implies that all bodies have the same spin angular momentum. This requirement is, as we have seen, a subsidiary condition that is consistent with the equations of motion that we have adopted. Of course, we have not yet explained why the subsidiary condition should be universally satisfied but this may perhaps be done later in connection with more extensive theories, such as, for example, those concerned with a causal interpretation of the Dirac equation and second quantization.

4. – An Illustration in Terms of Orbits in a Hydrogen Atom.

We shall now illustrate the causal interpretation of the Pauli equation in terms of the example of an electron in a hydrogen atom. This illustration

will also permit us to draw some interesting conclusions concerning the meaning of the quantization conditions in the hydrodynamic model with spin.

We begin by considering a stationary state, in which the spinor wave function α, is given by

$$\alpha = R(\boldsymbol{x}) \exp[-iEt/\hbar] \begin{pmatrix} \beta_1(\boldsymbol{x}) \\ \beta_2(\boldsymbol{x}) \end{pmatrix} = \tag{34}$$

$$= R(\boldsymbol{x}) \exp[i\psi(\boldsymbol{x}, t)/2] \begin{pmatrix} \cos\theta/2 & \exp[i\varphi/2] \\ i\sin\theta/2 & \exp[-i\varphi/2] \end{pmatrix},$$

where $R(\boldsymbol{x}) = \sqrt{|\alpha_1|^2 + |\alpha_2|^2}$ is the normalized spinor of eq. (7) and

$$\psi(\boldsymbol{x}, t) = -\frac{2Et}{\hbar} + \psi_0(\boldsymbol{x}), \qquad \text{with} \qquad \psi_0(\boldsymbol{x}) = [\psi(\boldsymbol{x}, t)]. \tag{35}$$

(Note that here $\psi(\boldsymbol{x}, t)$ refers to the *Euler angles* of the body which is a function of position, while α is the wave-function spinor).

We see then that in a stationary state the orientation angles θ and φ are in general functions of position, but are constants in time. However, the angle ψ of rotation about the first principal axis increases linearly with the time, when evaluated at a given position, $\boldsymbol{x}$.

If, however, we look at a particular moving body, then it will follow some orbit in space, in which the various Euler angles and spin directions may change with time. But the motion must be periodic, in the sense that after the body returns to its original position in space, the Euler angles θ and φ must return to their original values (plus perhaps some multiple of 2π for the Euler angle φ) while ψ must have changed by $-2E\Delta t/\hbar$ (plus a suitable multiple of 2π). If these conditions were not satisfied, then we could not have a stationary state in which the only change with time of the state of the fluid at a given point is at most a rotation of each body around its axis of symmetry.

To show how this wave function leads to quantization of angular momentum, we shall now evaluate the action integral, $T = \oint \boldsymbol{p}\cdot d\boldsymbol{x}$, where $\boldsymbol{p}$ is the momentum of the motion of the center of mass of the body in its periodic orbit. Expressing $\boldsymbol{p}$ with the aid of paper (A) eq. (23), we obtain

$$\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}(\nabla\psi + \cos\theta\,\nabla\varphi)\cdot \mathrm{d}\boldsymbol{x}.$$

Now $\mathrm{d}\boldsymbol{x}$ represents the change of $\boldsymbol{x}$ as we follow the moving body in its orbit. Let us then introduce the differential, $\delta\boldsymbol{x}$, which represents the change of $\boldsymbol{x}$ at a fixed instant of time. Now, from eq. (35), we have $\nabla\psi = \nabla\psi_0(\boldsymbol{x})$. Since neither θ, φ nor ψ_0 are functions of t, we can then replace $\mathrm{d}\boldsymbol{x}$ in the action

integral by $\delta \boldsymbol{x}$, obtaining

$$\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}(\nabla\psi + \cos\theta\,\nabla\varphi)\cdot\delta\boldsymbol{x} = \frac{\hbar}{2}(\delta\psi_0 + \cos\theta\,\delta\varphi) \tag{36-a}$$

and

$$\oint \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}\oint(\delta\psi + \cos\theta\,\delta\varphi)\,, \tag{36-b}$$

where on the right hand side of (36-*b*), the integration is carried out around a circuit in space taken at a particular time, but the same circuit as is made by the actual orbit of the body.

We must now find out how ψ_0 and φ change as we go a around the circuit. Obviously we must have $\Delta\psi_0 = 2n_1\pi$ and $\Delta\varphi = 2n_2\pi$ where n_1 and n_2 are integers, in general different from each other. But now we shall show that n_1 and n_2 must satisfy the further conditions that $n_1 + n_2$ and $n_1 - n_2$ shall be *even* numbers.

To prove this, we must first briefly review some of the topological properties of the rotation group ([5,6]). Consider, for example, a series of infinitesimal space displacements in our field of body orientations, which add up to give some finite displacement. Such a series of space displacements will lead to a corresponding series of infinitesimal rotations, which add up to give the total rotation needed to bring the body from the orientation that it had at the beginning of the series to the one that it has at the end. Now, if the series takes the form of a closed circuit, it is evidently necessary that the body finally return to its original orientation, so that, as we have already pointed out, an integral number of rotations of 2π must have taken place along the circuit. But now, let us suppose that we make the circuit smaller and smaller, permitting it to shrink continuously down to a certain point. The total rotation that takes place as we follow the circuit must then decrease continuously to zero. Otherwise, we should have a finite rotation connected with an infinitesimal displacement in space, and this would not be consistent with the requirement that the field of body orientations be continuous.

In the mathematical expression of the above requirement of continuity in the field of body orientations, the Euler angles do not provide a convenient parametrization, because when $\theta = 0$, singularities can appear in the para-

([5]) H. Weyl: *Theory of Groups and Quantum Mechanics* (New York, 1928). See p. 180.

([6]) F. D. Murnagham: *The Theory of Group Representations* (Baltimore, 1938). See p. 318.

metrization that do not correspond to any real physical singularity in the field of orientations. Indeed, when $\theta = 0$, the only quantity defined by the actual rotation of the body is the sum of the angles, $\varphi + \psi$, while the difference, $\varphi - \psi$ is completely arbitrary. Thus, if we deform our circuit past a point where $\theta = 0$, it is possible for φ and ψ separately to undergo large changes, even when the actual rotation (and therefore along with it the sum, $\varphi + \psi$) changes only by infinitesimal amounts. Hence continuous changes in the orientations of the bodies need not always be reflected as continuous changes in the Euler angles; and for this reason, there is no guarantee that as we shrink a circuit in a field of rotations down to a point, the corresponding circuit in the space of the Euler angles will also shrink down to a point. As a result, the expression of the requirement that the field of body orientations be everywhere continuous would be rather complicated if done in terms of Euler angles.

To avoid the kind of difficulty described above, we may parametrize the rotation group, not in terms of Euler angles, but rather, in terms of the Cayley-Klein parameters, which are (according to paper (A), eq. (8-b)):

$$b_1 = \cos\theta/2 \cos(\psi + \varphi)/2 , \qquad b_2 = \cos\theta/2 \sin(\psi + \varphi)/2 ,$$

$$b_2 = - \sin\theta/2 \sin(\psi - \varphi)/2 , \qquad b_4 = \sin\theta/2 \cos(\psi - \varphi)/2 .$$

For the Cayley-Klein parameters have the advantage that they provide everywhere a *locally* unique and continuous representation of the rotation group, in the sense that an infinitesimal rotation always implies a corresponding infinitesimal change in these parameters. To demonstrate this property of the representation, we note that the unit spinor (which is in a one-to-one correspondence with the Cayley-Klein parameters) transforms under the infinitesimal rotation, $\boldsymbol{\omega}\,\mathrm{d}t$, as $\delta\beta = i(\mathrm{d}t/2)(\boldsymbol{\omega}\cdot\boldsymbol{\sigma})\beta$. Thus, an infinitesimal rotation must *always* produce a corresponding infinitesimal change in β (and therefore in the C.K. parameters).

It must be remembered, however, that the unique character of the C.K. parameter representation applies only to *small* rotations. Indeed, with regard to the properties of the rotation group « in the large », this representation is two-valued. For corresponding to any given rotation, there always exist *two* sets of the b_i, one of which is the negative of the other. But even the two-valued character of the C.K. parameters in the large is significant; for it provides a mathematical description of the fact that the rotation-group is *doubly-connected*, a fact that is, as we shall see presently, very important in the determination of the changes of angle that are permissible in going around a circuit.

To see the meaning of the two-valued character of the representation of the rotation group in terms of the C.K. parameters (and therefore in terms

of the spinors), let us consider a circuit in space that leads to a continuous series of infinitesimal rotations of the body orientations. There is of course an ambiguity in the point in the C.K. space at which we must start, because, corresponding to the initial orientation of the body, there are two possible sets of values of the C.K. parameters. But after we have chosen one of these sets, then the path in C.K. space that corresponds to our series of rotations is determined uniquely because of the locally continuous character of the representation of the rotation group in terms of the C.K. parameters. Let us now consider a circuit leading to a rotation of 2π about some axis. It is clear from the definition of the b_i that such a rotation carries us from a given set, (b_1, b_2, b_3, b_4) of C.K. parameters to the corresponding set $(-b_1, -b_2, -b_3, -b_4)$. Thus, even though the body has come back to its original orientation, the C.K. parameters have not come back to their original values. Indeed, geometrically speaking, this rotation carries us only half way around the hyperspherical surface on which we must remain in the four dimensional space of the b_i. As in the case of a great circle on a sphere in three dimensional space, there is no way to *continuously* deform such a curve to a point *while keeping the endpoints fixed* (7). As a result, when we make a closed circuit in the field of body orientations, if the body rotates through 2π along this circuit, then as the circuit is shrunk down to a point, the total rotation that takes place along this curve will have to remain equal to 2π, no matter how small the circuit becomes, so that the field of body orientations cannot be continuous. On the other hand, if the circuit carries us through *two* rotations of 2π, we are brought back to the original values of the C.K. parameters; and in the space of these parameters, the corresponding path goes all the way around the hypersphere. Then, as in the case of a complete great circle on a sphere in three dimensional space, this curve can be deformed continuously to a point, while keeping the endpoints fixed. From this, we conclude that a circuit involving two rotations of 2π can be shrunk continuously to one having no rotation at all, while a circuit involving a single rotation of 2π cannot. More generally, this will happen wherever $\varphi+\psi$ and $\varphi-\psi$ change by even multiples of 2π on going around a circuit; or whenever n_1+n_2 and n_1-n_2 are *even* integers. Thus, we justify the conditions which we gave earlier, regarding the permissible changes of value that the wave function may suffer on going around a circuit.

We may illustrate the conclusions of the preceeding paragraph in terms of an example in which a given circuit carries us through two rotations of 2π

(7) The fixing of the end points corresponds to the fact that we are choosing a circuit that passes through a certain point in space. Since the orientation of the body is fixed at this point, the C.K. parameters cannot change at the endpoints of the circuit, no matter how this circuit is shrunk or otherwise changed.

in any specified direction. Now, as the circuit is shrunk, it would be possible for the rotation to become equivalent to two separate rotations of 2π, each carried out in a different direction; for in this case, the initial and final orientation of the body would still remain unchanged when the circuit was altered. Thus, a continuous deformation of the path in the C.K. space would take place, with the endpoints of the path held fixed; for the direction of one of the rotations of 2π could change continuously relatively to that of the other. But when we arrived at a path in which each rotation had an opposite sense, to that of the other then the two rotations would cancel each other; and we would have no rotation at all. Thus, we have given an example of how a circuit involving two rotations of 2π can be deformed *continuously* into one involving no rotation at all. As we have shown, however, this cannot be done with a circuit involving a single rotation of 2π.

We see then how the two-valued character of the C.K. parameters (and therefore of the spinors) reflects the topological connectivity properties of the rotation group. Of course, if we had to deal only with a single body, these topological properties would not be relevant, because a single rotation of 2π would, after all, bring the body to an equivalent orientation in space, so that at a given time, no physical difference could exist between a body that had suffered only a single rotation of 2π, and one that had suffered two such rotations. On the other hand, because we are dealing with a *field* of body orientations, the relative amounts of rotation suffered by bodies in different parts of the field become relevant; in the sense that if the orientation changes by an odd number of rotations of 2π along a circuit, then this circuit cannot continuously be shrunk to one with no rotation at all, while if it changes by an even number of rotations of 2π, it can so be shrunk. Thus, the two-valued character of the spinors (and of the C.K. parameters) describe physically significant properties of a field of body orientations. In this way, we obtain a reason why the basic quantum-mechanical theory of the electron which concerns itself with such a field of body orientations, should be expressed in terms of spinors.

We now apply the above conditions on the continuity of the angles of rotation to a stationary state of a hydrogen atom, which in the usual interpretation is described as having an orbital angular momentum, l, and a Z component of the total angular momentum (spin plus orbital) of $k = (l + \frac{1}{2})$. For this state, the wave function is [8]

$$\alpha = \frac{R(\boldsymbol{x})}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+k}\, P_l^{(k-\frac{1}{2})}(\theta') & \exp\left[i(k-\frac{1}{2})\varphi'\right] \\ \sqrt{l-k+1}\, P_l^{(k+\frac{1}{2})}(\theta') & \exp\left[i(k+\frac{1}{2})\varphi'\right] \end{pmatrix}, \tag{37}$$

([8]) See D. Bohm: *Quantum Theory* (New York, 1951); Chap. 17, eq. (73).

where θ' and φ' are the polar angles of the position vector, $\boldsymbol{x}$, of *the center of the body*, while $P_l^{(k-\frac{1}{2})}(\theta')$ is a suitably normalized associated Legendre polynomial.

Comparison with eq. (34) and (35) then indicates that

$$\frac{\psi_0}{2} = k\varphi' , \quad \varphi = -\varphi', \quad \mathrm{tg}\frac{\theta}{2} = \frac{\sqrt{l+k}}{\sqrt{l-k+1}} \frac{P_l^{(k-\frac{1}{2})}(\theta')}{P_l^{(k+\frac{1}{2})}(\theta')} . \tag{38}$$

Since $k = l + \frac{1}{2}$, and since φ' changes by 2π as we go around a circuit, we see that $\Delta\psi_0 = (2l+1)2\pi$ and $\Delta\varphi = -2\pi$. Thus, the topological relationship that $\varphi + \psi$ and $\varphi - \psi$ change by *even* multiples of 2π on going around a circuit is satisfied.

We are now ready to compute the action integral (75) for this case. To do this, we note that the velocity, $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\nabla\varphi)$, has only a component in the direction of φ'. Thus, the body moves in a circle, the plane of which is normal to the Z axis. From (77), we see that θ is constant along this circle. Thus we obtain

$$T = h\left(k - \frac{\cos\theta}{2}\right) = h\left(n + \frac{1}{2}(1 - \cos\theta)\right), \tag{39}$$

where n is an integer lying between 1 and -1.

The Z component of the *orbital* angular momentum of the body is

$$p_\varphi = \frac{T}{2\pi} = \hbar\left(k - \frac{\cos\theta}{2}\right). \tag{40}$$

The combined Z component of spin and orbital angular momentum is

$$p_\varphi + \frac{\hbar\cos\theta}{2} = \hbar k . \tag{41}$$

Thus, we find that the total angular momentum is just equal to that predicted in the usual theory. This angular momentum divides itself, however, between the spin and orbital angular momentum in accordance with the colatitude angle θ' of the radius vector of the body relative to the center of the atom (which determines θ through eq. (38)).

The result (39) for the action variable is analogous to that obtained from the Bohr-Sommerfeld quantum condition (of the « classical » quantum theory) in which even the feature of fractional quantum numbers is present, because these are often found to give better agreement with experiment than can be obtained with integral quantum numbers. There are, however, three important

differences from the Bohr-Sommerfeld theory. First, eq. (39) is an *exact* equation, and not an approximate one. Secondly, in the evaluation of J, the momentum, $\boldsymbol{p}$, must be calculated taking into account the quantum-potential and the spin energy, and not just the classical potential as was done in the Bohr-Sommerfeld theory. Finally, the action variable, J, for the particle motion alone depends in the location of the orbit. Only the sum of this action variable and the one corresponding to the spin angular momentum is quantized in a way that is independent of the location of the orbit.

We are now in a position to see how our model accounts for the quantization of the angular momentum. To do this, we first note that the action variable is proportional to $\oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}$ which is an integral representing the total *vorticity* of the fluid inside the circuit in question. Thus, we have obtained a simple physical interpretation of the action variable. Then, because of the relationship $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\nabla\psi)$, the connection (36) is established between this vorticity, and the changes of body angle, $\Delta\psi_0$ and $\Delta\varphi$, which occur as we go around a circuit. Finally, because we have a continuous field of spinning bodies, $\Delta\psi_0+\Delta\varphi$ and $\Delta\psi-\Delta\varphi$ must be even multiples of 2π. Thus, the quantization of vorticity and therefore of angular momentum reflects basically the requirement that we have a continuous and single-valued field of spinning bodies, whose orbital motion is coordinated to their spin motion in the way implied by the relationship, $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\,\nabla\varphi)$.

5. – On the Process of Measurement of the Spin.

We shall now indicate in general terms how the process of measurement of the spin is to be treated in the model given in this paper. To do this, we shall first review briefly a similar treatment of the analogous problem of a non-spinning particle (described by the *Schrödinger equation* and not by the Pauli equation) going around an atomic nucleus with orbital angular momentum, $\hbar$.

Now a common method of measuring the spin of an atom is to place the atom in a non-homogeneous magnetic field. This field then separates atoms according to their component of the angular momentum along the direction of the field. To show how this process of measurement is treated in terms of the causal interpretation of the quantum theory, we shall use a method developed in a previous paper [9]. Now the general initial wave function of the electron with orbital angular momentum of $\hbar$ is

$$\Psi = a_{-1}\Psi_{-1}(\boldsymbol{x}) + a_0\Psi_0(\boldsymbol{x}) + a_1\Psi_1(\boldsymbol{x})\,, \tag{42}$$

[9] D. Bohm: *Phys. Rev.*, **85**, 180 (1952).

where Ψ_{-1}, Ψ_0, Ψ_1 represent the wave functions of electrons with Z components of the angular momentum of respectively, $-\hbar$, 0 and $\hbar$, while a_{-1}, a_0, a_1, are the corresponding coefficients (in general complex) for the expansion of the wave function, $\Psi(\boldsymbol{x})$. The complete wave function for the combined system consisting of electron plus the atomic nucleus (whose coordinates we denote by $\boldsymbol{y}$) is then

$$f_0(\boldsymbol{y})(a_{-1}\Psi_{-1}(\boldsymbol{x}) + a_0\Psi_0(\boldsymbol{x}) + a_1\Psi_1(\boldsymbol{x})) , \tag{43}$$

where $f_0(\boldsymbol{y})$ represents a wave packet describing the fact that the atomic nucleus is fairly well-localized in space.

Now when we apply a magnetic field, $\mathfrak{H}$, (the direction of which we take to be that of the Z axis) then each of the three parts of the wave function described above will begin to oscillate at a different frequency. Moreover, if the magnetic field is inhomogeneous, the three parts will begin to separate in space. Indeed, after some time, the wave function will be transformed into

$$\Phi = f_0(\boldsymbol{y} - \Delta)a_{-1}\exp[i\alpha_{-1}]\Psi_{-1}(\boldsymbol{x}) + f_0(\boldsymbol{y})a_0\exp[i\alpha_0]\Psi_0(\boldsymbol{x}) + \\ + f_0(\boldsymbol{y} + \Delta)a_1\exp[i\alpha_1]\Psi_1(\boldsymbol{x}) , \tag{44}$$

where α_{-1}, α_0, α_1 are changes of the phase angles which have resulted from the effects of the magnetic field, and Δ is the motion of the nucleus under the action of this field. Now eventually Δ becomes much bigger than the width of the wave packet $f_0(\boldsymbol{y})$, so that the three parts of the wave function cease to interfere with each other, and obtain a classically distinct separation. When this happens, then as has been shown (9), the particle-like inhomogeneity associated with the electron must have entered one of the separate packets. The probability that it is any one of them is given by

$$P_{-1} = |a_{-1}|^2, \qquad P_0 = |a_0|^2, \qquad P_1 = |a_1|^2 . \tag{45}$$

Thereafter, the other packets play no role, so that they can be neglected.

We see then that what really happens in a measurement is that the measuring apparatus provides a general environment (in this case, the inhomogeneous magnetic field) in which the wave function is transformed from whatever it may have been initially into an eigen-function of the « observable » that is being measured (in this case, the Z component of the angular momentum).

Thus, in the case under discussion, the actual orbital angular momentum of the electron can vary continuously but only certain stable quantized values are possible, which can exist indefinitely without change, in the environment

supplied by the inhomogeneous magnetic field. Which of the three possible stable values will actually be obtained is not determined in an individual case, if we merely specify the initial wave function, $\Psi(\boldsymbol{x})$. To determine, the actual result, we should have to specify also the initial location, $\xi(t_0)$, of the particle, and the general irregular motions in the Ψ field which lead to a statistical behavior that is described by the probability distribution, $P = |\Psi|^2$ for the particles. But as we have seen, the probability of obtaining any particular result in a statistical aggregate of cases is determined in terms of the coefficients of the initial wave function with the aid of eq. (45).

In the causal interpretation of the quantum theory, a typical measurement process of the kind that can be carried out in connection with the atomic level is therefore not really a measurement of the detailed properties of the underlying system (which is assumed to consist of the Ψ field plus the particle that moves in it). Rather, it represents a kind of statistical response to certain over-all properties of the Ψ field and of the particle. In this sense, the « observables » are rather analogous to the pressure and temperature of macroscopic physics or to the macroscopic variables used in hydrodynamics, since these likewise do not describe any detailed properties of the underlying molecular motions, but rather general statistical properties of the system as a whole.

Let us now consider how this theory of measurements could be extended to the Pauli spin theory. To do this, we should need a theory of the many-body problem, corresponding to our theory of the many-body-Schrödinger equation ([9]), because as we have seen, we have had to discuss the interaction of the electron with the atomic nucleus in order to develop a theory of measurements. Such a theory is now being developed in terms of an extension of our model to include second quantization; and preliminary results suggest that such an extension will be possible, although many of the details remain to be worked out. For the present, however, we shall merely state that it appears that the theory of measurements can be carried out for the spin theory in a way that is essentially the same as what has been done for the Schrödinger equation without spin. For example, in the case of a spinning electron in an atom in an « s » state, there will be two possible basic solutions, one corresponding to spin up, and the other to spin down, with the spin measured in any desired direction. Hence, the initial wave function can be expressed as

$$\Psi = a_+\Psi_+(\boldsymbol{x}) + a_-\Psi_-(\boldsymbol{x}) \,.$$

Then, as in the case that we have already discussed of a particle of angular momentum, $\hbar$, obeying Schrödinger's equation, the effect of the inhomogeneous magnetic field will be to transform this wave function either into Ψ_+ or into Ψ_-, with respective probabilities, $P_+ = |a_+|^2$, and $P_- = |a_-|^2$, of obtaining

either result. Hence, as in the Schrödinger theory, the appearance of quantized possible values for the spin in a measurement will be the result of a reaction to the magnetic field of the system as a whole, (consisting of fluid with spinning bodies, and the particle-like inhomogeneity, described in Sec. **1**, which is the counter part of the particle appearing in Schrödinger's equation). Thus, what is obtained in what is now called a measurement of the spin will have no direct and simple relationship to the spins of the bodies constituting the fluid. Indeed, as we have seen, the components of the latter can vary continuously, but nevertheless, in a magnetic field, the overall motion of the whole system is such that the total angular momentum eventually settles down either to $\hbar/2$ or to $-\hbar/2$. (We may make here an analogy to certain kinds of classical non-linear oscillators, which after being disturbed eventually settle down to one of a number of possible stable modes of oscillation). It is clear then that the spin as it is now measured should be considered as a higher-level property, having a relationship to the assumed spinning bodies that is somewhat analogous to the relationship between macroscopic variables, such as pressure and temperature, and the underlying atomic variables.

6. – Summary and Conclusions.

In this paper and in the previous paper (A) we have developed a model for the Pauli equation in terms of a fluid composed of spinning bodies, which contribute an « intrinsic angular momentum » to the total angular momentum of the system. This model has the property that if the bodies are at any time all spinning with their angular momenta parallel to their principal axis of symmetry, then they will continue to satisfy this condition for all time. On the other hand, it is possible for the angular momentum S to have a general orientation; and in this case, the component of S normal to the principal axis will turn with an angular velocity, ω, that depends on how fast the body happens to be spinning and in the torques acting on the body. For processes with characteristic times, $\tau \gg 1/\omega$, the component of S normal to the principal axis will average out to zero, and the Pauli theory will provide a good approximation. But for processes in which τ is of the order of $1/\omega$ or less, the Pauli equation will no longer apply, and the full general set of non-linear equations will be needed. This means that our model already implies the possibility of a break down in connection with sufficiently high frequencies, and therefore with sufficiently high energies, of the whole general scheme connected with the usual interpretation of the quantum theory, which is based in an essential way on the assumption that the fundamental equations of the theory will *always* be linear.

In connection with our discussion of the theory of measurements in Sec. **5**,

it was seen that the spinning bodies of which our fluid is assumed to be constituted are not identical with the spin « observables » of the usual quantum theory, but that rather, they constitute a lower level, in terms of which the spin « observables » are determined as overall and in general statistical properties of the fluid. For example, the characteristic quantized way in which the spin angular momentum manifests itself at the atomic level was seen in Sec. 4 to follow from conditions of single-valuedness applying to the motion of the fluid as a whole, which are such that even though the spin motions of the bodies are continuous, the overall motion has certain discrete possible stationary values for the angular momentum.

In sum, then, it may be said that we have, for the case of the Pauli equation, *explained* the quantum theory in terms of the motions of new entities existing at a sub quantum-mechanical level. We call the new level « sub quantum-mechanical » because the laws of quantum-mechanics do not apply there. Rather, the laws of quantum mechanics emerge as overall and statistical relationships arising on the basis of the lower level laws, as for example, the laws of ordinary hydrodynamics arise on the basis of lower level laws governing the atomic motions.

Naturally, to make an explanation of the quantum theory possible, we have had to postulate something, viz, the fluid composed of spinning bodies (since without assuming something we can never explain anything). It may then be asked what we have gained by making such a postulate. First of all, we have gained the possibility of seeing in a rational way how all of the phenomena of atomic physics could be connected by means of a set of general causally determined motions, so that we do not have to regard atomic phenomena as mysterious processes which take place in a way that could never even be conceived of. (In this connection, let us recall that there has existed a widespread general impression that to obtain such a rationally understandable explanation of quantum phenomena in general and of the electron spin in particular would be impossible.) Secondly, whenever one obtains a rational explanation of a wide range of phenomena past experience in science has shown that this explanation generally suggests fruitful new avenues of approach to problems, which would not even have been suspected if the phenomena had not been thus explained, but rather had simply been accepted as things that « just happen » for no particular reason whatever. For example, the atomic theory, suggested originally by the effort to explain the laws of chemical combination and the gas laws in terms of the properties of atoms, was eventually able to explain many new kinds of phenomena (e.g., Brownian motion, viscosity, gaseous discharges, etc.) and suggested important new directions of research (e.g., Rutherford scattering, electron theory of metals, etc.). As for the question of wether the assumption of a sub quantum-mechanical level will eventually prove to be fruitful in a similar way, this can of course be

answered definitively only in the future. Nevertheless, one can already see good reasons why this approach may be on the right track, even if, perhaps, not correct in all of its details. Thus, the characteristic new phenomena of modern high energy physics is the appearance of a whole host of « elementary » particles, which can be « created », « destroyed » and transformed into each other. The very fact that these processes of creation, destruction and transformation are possible suggests strongly that the so-called elementary particles are not really elementary, but rather, that they arise on the basis of motions of new kinds of entities that are still more fundamental. Thus, what is suggested is a *new level*, below that of the « elementary » particles, out of which these particles arise as some kind of moving structures.

Now, we have already seen that to explain the quantum theory causally, we have already had to postulate a sub-quantum mechanical level, out of the motions of which the usual quantum mechanical properties of things arose as overall characteristics (e.g., quantization). Now, as long as the basic equations governing the system are *linear*, nothing new can arise in these overall characteristics, not already treated in the well-known solutions of the Pauli equation. But as we have seen, it is just in connection with sufficiently high energy processes that the equations of our model can become *non-linear*. Now, it is well known that non-linear equations have, in general, many modes of stable motion. Each of these modes would manifest itself at the atomic level in connection with new rules for quantization and for the determination of other overall properties of the system, which we would interpret in terms of the appearance of a new kind of « particle ». Thus, the way is opened up for a treatment of the processes of creation, destruction, and transformation of elementary « particles », as well as for a calculation of which kinds of « particles » can exist, and of what some of their properties are, since the new « particles » could correspond to new modes of overall motion of the underlying fluid.

Of course, we do not believe that a model based on an explanation of the Pauli equation will really be adequate for the purposes described in the previous paragraph because it is not relativistic. A model based on an explanation of the Dirac equation (and better still with second quantization) should however give a much more accurate treatment than would be possible with the model given in this paper. Present work indicates that models can already be found which reproduce most of the features of the Dirac equation and many of those of second quantization. The completion of this work would then lay the foundation for an attack on the properties of the new level, including those connected with the creation, destruction and transformation of elementary particles. In any case, it is clear that new directions of investigation could thus be opened up, going outside the framework of theories that fit into the current general scheme of the quantum theory.

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